

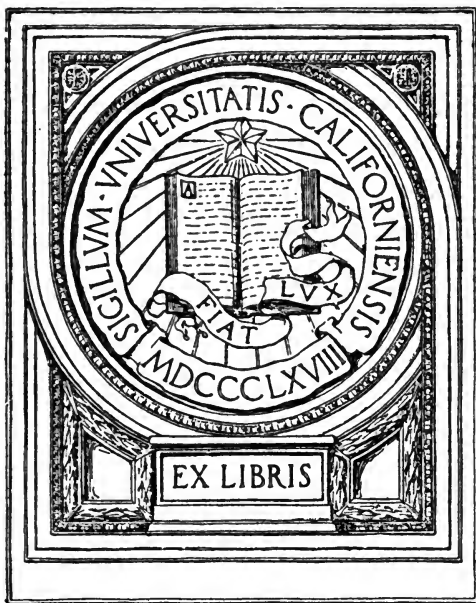
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RECOMMENDATIONS.

From Rev. C. H. Alden, Principal of the Philadelphia High School for Young Ladies.

MR. GREEN :

Dear sir,—I am greatly pleased to find, that, in your “Gradations in Algebra,” recently published, you have rendered the elements of that department of the Mathematics so attractive to the young student. It has long surprised me that this interesting method of analysis has been so entirely excluded from our common schools. With your valuable aid, however, this neglect can no longer find a suitable apology. I anticipate the introduction of your excellent work into both our private primary and public schools.

Very respectfully,

C. H. ALDEN.

RICHARD W. GREEN :

Sir,—I have examined your “Gradations in Algebra” with much care, and have no hesitation in saying that in my opinion you have fully attained your object in forming “an easy introduction to the first principles of algebraical reasoning,” and of furnishing “in the same course a popular exposition of the most important elements of arithmetic.” To write a work on abstruse science, adapted to the comprehension of youth, is a work of extreme difficulty. I feel that I offer high but deserved and just praise when I say that in the book before me you have fully succeeded.

Yours, &c.

JAS. RHODES,

Principal of N. W. Grammar School, Philad.

Dear sir,—Your “Gradations in Algebra” appears to me to supply what has long been wanted by those who are commencing that study, viz. an initiatory text-book at once small, clear, well arranged, and comprehensive. It meddles with nothing beyond the capacity of a school-boy. It is well supplied with such examples as show the learner the use of what he is studying. I would rather put your book in the hands of a beginner than any other work on the same subject now in common use.

Yours very respectfully,

JNO. W. FAIRES.

I examined "Green's Gradations in Algebra," and was so well pleased with it, that I introduced it into our school. I have made use of it one term, and can bear testimony that it has equaled my expectations. I consider it as occupying an important link between arithmetic and literal algebra. In my judgment, we have long needed such a work.

JOHN D. POST,
Teacher of Mathematics in Hartford Grammar School.

From Porter H. Snow, A. M., formerly Principal of the Hartford Centre School, and now Principal of Brainard Academy, Haddam, Conn.

I have examined critically your "Gradations in Algebra," and find it admirably adapted to fulfil its design. There is nothing so good on the subject in the English language for a student to read before entering college. All the mystery that clothes mathematics during the early part of a collegiate course of study will be solved and made plain, if the student will spend a few weeks on this introduction to algebra. It is also a fine work for classes in schools. And I would recommend its introduction into primary and higher schools in preference to any treatise on algebra I have ever seen. It is also the only thing of the kind I have met with, suitable for a student to peruse without an instructor.

York, Pa., Nov. 30th, 1841.

After a careful examination of Mr. Green's "Algebra," we have no hesitation in expressing our decided approbation of the work. The clearness of the explanations, the judicious and systematic arrangement of the parts, and the gradual manner in which the student is led on from the first principles to the more difficult parts of the science, render it, in our view, preferable to any we have seen for primary instruction.

DANIEL KIRKWOOD,
Teacher of Mathematics, York Co. Academy.

D. M. ETTINGER,
Principal of the High School, York, Pa.

The great excellence of this work is, that it brings the pupil on gradually. Only one difficulty is presented at a time; and upon this, explanations and examples are multiplied till the pupil becomes so familiar with the subject that he almost wonders where the difficulty is at which he first stumbled. The examples are numerous; the few rules (and they are enough) are admirably expressed, in plain, concise language. On the whole, we may say, that if a youth is to begin the study of algebra, this should be his first book

Extract from the North American.

J. Cajon

GRADATIONS

IN

ALGEBRA,

IN WHICH

THE FIRST PRINCIPLES OF ANALYSIS

ARE

INDUCTIVELY EXPLAINED.

ILLUSTRATED BY

COPIOUS EXERCISES,

AND MADE SUITABLE FOR PRIMARY SCHOOLS.

BY

RICHARD W. GREEN, A.M.,

AUTHOR OF ARITHMETICAL GUIDE, LITTLE RECKONER, ETC.

PHILADELPHIA:

PUBLISHED BY E. H. BUTLER & CO.

1850.

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{ Chamber of the Controllors of Public Schools,
First School District of Pennsylvania.

PHILADELPHIA, November 15, 1849.

At a meeting of the Controllors of Public Schools, First District of Pennsylvania, held at the Controllors' Chamber, on Tuesday, November 13, 1849, the following Resolution was adopted:—

Resolved, That Green's Algebra be introduced, as a Class-Book, into the Grammar Schools of the District.

From the minutes,

R. J. HEMPHILL, *Secretary*.

Secretary's Office, Harrisburg, Feb. 4, 1843.

MR. RICHARD W. GREEN:

SIR,—I have examined the work prepared by you for the use of primary schools, entitled "Gradations in Algebra." It meets with my approbation, and I am much pleased with its design and arrangement. I consider it admirably calculated to aid the pupil at the commencement of the science, also to give him a general knowledge of it.

It is, in my opinion, well adapted to the use of common schools, where there are students who have not the time or means of consulting more extended treatises upon the subject. I hope it may be introduced generally into our schools throughout the state.

I am yours respectfully,

A. V. PARSONS,

Superintendent Common Schools.

Entered according to the Act of Congress. in the year 1839, by

RICHARD W. GREEN,

in the Clerk's Office of the District Court of the Eastern District of
Pennsylvania.

CAJORI

✂ For the convenience of those teachers who may wish to adopt the author's plan of using this book, a key has been prepared containing the solutions of all the examples and problems. It will also save the teacher much time when he wishes to find for his pupil the errors in his work.

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P R E F A C E.

THE object of the author, in composing this treatise, was to form an easy introduction to the first principles of algebraical reasoning; and also to embrace, in the same course, a popular exposition of the most important elements of arithmetic. And he believes that he has been enabled to combine the rudiments of both, in such a manner as to make the operations of one illustrate the principles of the other.

In order that this method of treating the subject might preserve its chief advantage, especially in the initiatory course of the study; the work has been divided into two parts—*Numeral Algebra* and *Literal Algebra*.

In *Numeral Algebra* I have treated of the several primary arithmetical operations; first making them intelligible to very young pupils, and then exhibiting them under the algebraical notation. By this means, as every lesson in algebra is immediately preceded by corresponding numerical exercises, the transition from one to the other has been made so trifling, that the pupil will feel at each step that he has met with nothing more than what he has already made himself familiar with in a different dress.

Besides, as algebraical operations require the exercise of abstraction in a greater degree than the pupil is supposed to be accustomed to, I have taken care that the exercise on each of the fundamental rules, shall be followed by a selection of problems to be solved by equations.

As mathematical questions of this kind are always pleasing to young pupils, this arrangement will serve to impart an interest to the study at the commencement, and also to preserve a taste for it through the whole course.

Indeed, this part of the work is the most important for those pupils who do not intend to pursue the mathematical sciences. For, it is in such exercises, that the mind is trained to investigate the relations of one thing to another, and to conduct its reasonings in a clear and forcible manner.

Under the head of Literal Algebra, I have repeated, in a more strictly *algebraical form*, the principles which have been explained in the preceding part of the work ; and have shown some of their uses by applying them in the deduction and demonstration of several abstruse operations on numbers.

But the great peculiarity of the book is, that it habituates the *speech* and the *ear* to mathematical language. In any study, it is necessary for beginners to receive such a course of training as will imprint upon their minds each new idea, as soon as it is apprehended. Learners in the mathematics, especially, are accustomed to forget soon, both the names and the use of the signs ; and also the arrangement of the several steps in the solution of their problems. On this account I have required the pupil *always to repeat verbally* the operation that he has performed ; taking care to omit no part of the work that would hinder an auditor from understanding the reason for the several steps, and consenting to the just conclusions of the answer which has been obtained. It has been found by experience that this simple device enables the young pupil to acquire the science very easily ;

and while it impresses his lessons indelibly upon his memory, it also develops his genius, rectifies his inventive faculties, and imparts, as it were, a mathematical form to his mind; so much so, that he is generally capable of pursuing the subject afterwards by himself.

In order to accomplish this end more perfectly, I have swelled the number of examples beyond the ordinary limits. These should be thoroughly mastered as the pupil proceeds. There must be no smattering in the beginning of a science if the learner is to continue the study. The author has found by long experience, that a book is sooner finished when each part has been made familiar to the mind, than when it has been superficially attended to.

With regard to the arrangement of the several divisions, I have been careful to introduce first those principles that will be the most easily apprehended; and afterwards such others as would most naturally arise from the former if the study were entirely new. This method appears to be the best adapted for teaching the rudiments of a science; although in a succeeding text book, it is necessary that the arrangement of the several parts should be more systematic. On this account the advanced scholar must not be surprised to find in the middle of the book, what he has been accustomed to see near the beginning of other treatises. However, so much regard to a regularity of arrangement has been attended to, that the pupil will be assisted by the associations of method, both to understand and to remember.

As the author wishes to bring the study of Algebra within the reach of common schools, he has endeavored to prepare this work, so that it may be studied by pupils who are

not already adepts in arithmetic. And it is believed that such learners will not fail of obtaining, by a perusal of it, a full understanding of vulgar fractions, roots and powers, proportion, progression, and other numerical operations that are generally embraced in arithmetical treatises.

R. W. G.

ADVERTISEMENT.

The foregoing is the Preface of the author's "Inductive Algebra." The first 192 pages of that book have been published in this form, in order to afford a cheap manual for those classes that do not wish to study beyond *Quadratic Equations*. In the present state of education, so much of Algebra should be studied by every pupil in our common schools

PART I.

NUMERAL ALGEBRA.

PRELIMINARY REMARKS.

§ 1. ALGEBRA is a method of arithmetical computation, in which the calculations are performed by means of letters and signs.

§ 2. When the answer of an arithmetical question is to be found by algebra, *we first represent that answer by a letter*; because we *can use that letter* in our calculations, in the same manner as if it were the true answer.

We will explain by a few examples.

Example 1. I have 27 apples which I wish to give to two children, John and Mary, in such a manner that Mary shall have twice as many as John. The following will be our method of finding how many each shall have.

I wish to give to John a *certain number*, and to Mary *twice that number*. Therefore, instead of saying, I will give the 27 apples to John and Mary, I may say, I will give away a *certain number*, and then again *twice that number*.

Now, a *certain number*, and then *twice* that number, make, in all, *three times* that certain number.

But, I give away 27 apples; and hence I know that

Three times that certain number is just as much as 27.

Therefore, that number itself is $\frac{1}{3}$ of 27, which is 9.

Hence, John has 9 apples; and Mary has twice as many, which is 18.

§ 3. If, in the operation which we have just performed, we use the letter N , instead of the words, *a certain number*, our work will be somewhat abridged. Thus:

John will have a certain number, which is N .

Mary will have twice as much, which is *twice* N .

And both together will have *three times* N .

But, in the question, we said both shall have 27

Therefore, *three times* N is the same as 27: and *once* N is $\frac{1}{3}$ of 27, which is 9; which is John's share.

§ 4. About the year 1554, Süfelius, a German, introduced the sign $+$ for the words *added to*; and called it *plus*. Ever since that time, $+$ has been used to signify that the quantity after it has been added to the quantity before it. Thus, $2 + 6$, is read 2 *plus* 6, and signifies 2 with 6 added to it.

Example 2. A brother told his sister that he was 4 years older than she; and that *his* age and *her* age, when put together, made 18 years. What was the age of each?

By the statement, *his age* was the same as her age with 4 years put with it; because it was 4 years *more* than hers.

Therefore we will represent the sister's age by A

And his age, which is 4 years more, will be $A + 4$

Both ages together, will be A and $A + 4$, or $A + A + 4$

Which, because the A 's can be put together, is *twice* $A + 4$

But both their ages together made 18

Therefore *twice* $A + 4$ is the same as 18,

And *twice* $A + 3$ is the same as 17,

And *twice* $A + 2$ is the same as 16,

And *twice* $A + 1$ is the same as 15,

And *twice* A alone is the same as 14,

And A , or the sister's age, is $\frac{1}{2}$ of 14, which is 7.

§ 5. About the year 1550, John Scheubelius, a German, introduced the following practice. Instead of writing 2 *times* A , or 3 *times* A , or 4 *times* A , &c., he wrote $2A$, $3A$, $4A$, &c.

Example 3. Two men, A and B, owe me \$270; and B's debt is twice as much as A's. How much does each of them owe me?

We will represent A's debt by A .

Then, B's debt will be twice as much, or $2A$.

And both debts put together, make $A + 2A$.

But both, when put together, are equal to \$270.

Therefore, $A + 2A$ is the same as \$270.

Putting the A 's together, $3A$ is the same as \$270.

Then, one A is $\frac{1}{3}$ of \$270, which is \$90.

A's debt is \$90; and B's debt is twice as much, or \$180.

§ 6. In 1557, Dr. Recorde, an Englishman, introduced the sign $=$, which we call *EQUALS*. It is used instead of the expression, *is the same as*, or, *is as much as*, or, *is equal to*. Thus, $2 + 6 = 8$, is read, *2 plus 6 equals 8*.

Example 4. B's age is three times A's; and C's age is double of B's; and the sum of all their ages is 70 years. What is the age of each?

Let us represent A's age by the capital A .

Then, B's age will be three times as much; that is, $3A$.

And C's age will be twice $3A$, which is $6A$.

Then, all their ages put together will be $A + 3A + 6A$.

But all their ages put together is 70 years.

Therefore, $A + 3A + 6A = 70$.

Putting the A 's together, $10A = 70$.

Dividing by 10, $A = 7$, which is A's age.

A's age $= 7$ years.

B's is three times as much; or 21 years.

C's is twice B's; or, 42 years.

Proof, $\frac{70}{70}$

§ 7. We very often wish to state that we have made a quantity to be less. This was formerly done by using the word *MINUS*. Thus, if I wished to say that your age is 6 years less than mine; I may say, Your age is equal to mine

when 6 years are subtracted from mine ; or, in fewer words, your age is equal to my age, *minus* 6 years.

§ 8. Stifelius introduced the sign — for *minus* ; so that $20 - 6$, is read, 20 *minus* 6 ; and signifies, 20 with 6 *subtracted* from it. Thus, $20 - 6 = 14$.

Example 5. At a certain election, 548 persons voted ; and the successful candidate had a majority of 130 votes. How many voted for each ?

Let us represent the number of votes received by the *successful* candidate, by the letter A . Then, as the unsuccessful candidate received 130 votes less, his number may be represented by $A - 130$.

Both candidates have $A + A - 130$

Then, $A + A - 130 = 548$

Putting the A 's together, $2A - 130 = 548$, which means, $2A$, after 130 is subtracted from it, is equal to 548

Now, if we had a barrel of water, minus a quart, and wanted a full barrel, we should add the quart. In the same manner, as we have $2A$ minus 130, we must add the 130 to get the complete $2A$.

Therefore, adding the 130 to both sides, $2A = 678$

Dividing by 2, $A = 339$.

The successful candidate had 339.

The unsuccessful one, 130 less = 209.

Proof, 548.

§ 9. *Des Cartes*,* a Frenchman, who wrote about 1637, used the *last letters of the alphabet* ; namely, x, y, z, u , &c., to denote the *unknown quantities*. And this is now the practice of all mathematicians.

Example 6. It is required to divide \$300 among A, B, and C ; so that B may have twice as much as A, and C may have as much as A and B together.

* Pronounced *Da-Cart'*.

Let us represent A's share by x . Then, B's share will be $2x$; and C will have as much as both put together, which is $3x$. Then,

$$x + 2x + 3x = 300.$$

Putting the x 's together, $6x = 300.$

Dividing by 6, $x = 50.$

Therefore, A's share is \$50.

B's is twice as much, or \$100.

C's = as much as both, or \$150.

Proof, \$300.

From what we have shown thus far, it may be seen that Algebra is a kind of language, made up with letters and signs in such a manner that all the reasonings which are necessary for the solution of a question, may be contained in a very small space, and be perceived with great facility.

§ 10. As we sometimes wish to speak of particular parts of our calculations, mathematicians have given the name *term* to any quantity that is separated from others by one of the signs $+$ or $-$. Thus, in the last example, and first line of the operation, the first x is the first term, the $2x$ is the next term, and the $3x$ is the next term; and the 300 is the last term.

§ 11. When a figure is put before a *letter*, to denote how many times we take the quantity which that letter stands for, the *figure* is called a *co-efficient*.* Thus, in the term $2x$, 2 is a co-efficient of x ; in the term $3A$, 3 is a co-efficient of A .

§ 12. It must also be understood, that a letter *without* any number before it, has 1 for its co-efficient. Thus, x represents $1x$; a is the same as $1a$, &c. The 1 is omitted because it is plainly to be understood.

§ 13. When any algebraic term stands by itself, it is called a *simple quantity*. Thus, x is a simple quantity; $2x$ is a simple quantity; 12 is a simple quantity, &c.

* This name was given by Franciscus Vieta, about 1573.

§ 14. Quantities that consist of more than one term are called *compound quantities*. Thus, $a + b$ is a compound quantity; so is also $a - b$, and $x + 7$, and $x - 7$ and $a + x - 7 - b$, &c.

§ 15. When any algebraic quantity begins with the sign $+$, it is called a *positive quantity*; as, $+a$, $+3a$.

§ 16. When any algebraic quantity begins with the sign $-$, it is called a *negative quantity*; as, -6 , $-5a$.

§ 17. In algebra, the *perfect* representation of any simple quantity requires both the specified sum, and either the sign $+$, or the sign $-$; as, $+5$, -40 , $+x$, $-3x$.

§ 18. But, when a *positive* quantity stands by itself, or when it is the first term of a compound quantity, the sign that belongs to it is generally omitted on paper, and also in our reading; as, x , 2 , $a + b$, $x - y$.

§ 19. Therefore, when a simple quantity, or the first term of a compound quantity, does not begin with a sign, we say that the sign $+$ is understood. That is, we think of the quantity the same as if $+$ was before it.

§ 20. In reading compound quantities, the pupil must be careful to join the sign to the term that is *immediately after* it. Thus, the following quantity must be read *a; plus b; plus 8; minus 4; plus 2a; plus 6; minus 3b*:
 $a + b + 8 - 4 + 2a + 6 - 3b$.

§ 21. When a quantity is expressed by figures, it is called a *numeral* quantity. When it is represented by a letter, it is called a *literal* quantity.

SIMPLE EQUATIONS.

§ 22. The most general and useful application of algebra, is that which investigates the *values of unknown quantities* by means of *equations*.

§ 23. An equation is an expression which declares one quantity to be equal to another quantity, by means of the sign $=$ being placed between them.

Thus, $5 + 3 = 8$, is an equation, denoting that 5 with 3 added to it, equals 8. Also, $4 - 1 = 3$, and $3 + 2 - 1 = 4$, and $8 - 2 = 5 + 1$, are equations, each denoting that the quantity on one side of the $=$, is equal to that on the other side.

§ 24. The whole quantity on the left of $=$ is called the *first member of the equation*; and all on the right of $=$ is called the *last member of the equation*.

In order to be a member of the equation, it is of no importance whether the quantity is simple or compound. Thus, in the equation $x = 4 + a - b - 18$, x is the first member, and $4 + a - b - 18$ is the last member. And in this case, the first member represents just as great a quantity as the last.

§ 25. In order that an equation may be such that it will enable us to find the value of an unknown quantity by it, it must contain some quantity that is already known. And then we can find the value of the *unknown* quantity, by making it stand by itself on one side of $=$, and all the known quantities on the other side.

§ 26. But it is generally the case that the known and unknown quantities are mingled together. And then we are to alter the arrangement of the terms, so as to bring the unknown quantity by itself; and this must be done without destroying the equality of the sides. This operation is called *solving or reducing the equation*.

§ 27. When the *known* quantities are represented by numbers, the equation is called a *numerical* equation. It is to this kind of equations that the first part of this treatise relates.


§ 28. In solving equations, the different arrangement of the terms is brought about by addition, subtraction, multiplication, division, &c., as the case may require.

SECTION I.

SIMPLE ALGEBRAICAL OPERATIONS.

I. ADDITION AND SUBTRACTION OF SIMPLE QUANTITIES.

§ 29. In algebra, *simple quantities are added by writing them down one after another; being careful to retain the signs that are expressed or understood.* Thus, we add 9 to x , by writing them $x + 9$; because, when those quantities were alone, $+$ was understood before 9. We add -9 to x , by writing them $x - 9$.

 The pupil must understand that x stands for some number; but it is often the case that we do not know what that number is. For it may stand sometimes for one number, and sometimes for another.

§ 30. It will readily be seen, that it is of no consequence which quantity is put first; for $9 + 7$ is the same amount as $7 + 9$.

§ 31. *One positive simple quantity is subtracted from another simple quantity, by writing down both quantities one after another, and changing from $+$ to $-$ the sign before the quantity which we subtract.* Thus, we subtract 4 from x , by writing them $x - 4$; which is read x minus 4; and *not* 4 from x .

Questions. What is algebra? See § 1. In what manner do we use letters in our calculations? 2. What is said of the sign $+$? 4. $-$? 7. $=$? 6. What does *term* signify in algebra? 10. What is a *simple* quantity? 13. How are simple quantities added in algebra? 29. What is requisite for the perfect representation of an algebraic term? 17. Are all terms thus represented? What is understood in such cases? Is the sign $-$ ever understood? In addition, which quantity must be put first? 30. What particular number does a letter stand for in algebra? Why do we use letters in algebra? 2. How is one simple quantity subtracted from another? 31. Which quantity must have $-$ before it?

§ 32. When two simple quantities have been added together, or one simple quantity subtracted from another, the answer will consist of more than one term, and become a compound quantity.

§ 33. It sometimes happens, that in those compound quantities which are made by adding or subtracting, there are two or more terms of the same kind; that is, such terms as do not differ at all, or differ only in their *numeral* co-efficients; such as $x + 2x$; $4a - 2a$; $5x - 3x + x$. Such quantities are called *similar* quantities.

§ 34. When, in any compound quantity, there are two or more terms of the same kind, they may be *united* by performing *arithmetically* the operation which is expressed by the signs belonging to them. Thus, $x + 2x$ is united into $3x$; $5a - 3a$ is united into $2a$; $5x - 3x + x$ is united into $3x$.

§ 35. Numeral quantities may be united in the same manner. Thus, $4 + 3$ may be united into 7 ; $9 - 4$ may be united into 5 ; $6 - 2 + 5$ may be united into 9 .

Examples. The pupil may unite the following quantities :

| | | |
|---------------------|--------------------|--------------------------|
| 1. $a + 3a + 5a$. | 5. $b + 6b - 3b$. | 9. $3x + 2x + 8 - 4$. |
| 2. $x + 5x + x$. | 6. $7a - 4a + a$. | 10. $10a - 3a + 7 - 2$. |
| 3. $10x - 4x - x$. | 7. $4 + 8 - 3$. | 11. $6a + 17 - 3a - 7$. |
| 4. $a + 11a - 2a$. | 8. $5 - 2 + 7$. | 12. $14 + 5x - 3x - 5$. |

2. GENERAL RULE FOR UNITING TERMS.

§ 36. 1st. When *similar* quantities have the *same* sign, select one kind, and find the sum of the numeral co-efficients;

Questions. What is a compound quantity? 14. What operation will produce compound quantities? What are *similar* quantities? What is a co-efficient? 11. Have all terms a co-efficient expressed? 12. What may be done with similar quantities? When a sign is between two quantities, to which quantity does it belong? 20. How does a literal quantity differ from a numeral? 21. What is a *positive* quantity? 15. What is a *negative* quantity? 16. What is the first part of the rule for uniting terms? What is the second part of the rule for uniting terms?


and, preserving the sign, annex the common letter. Then select another kind, and proceed in the same manner with that.

EXAMPLES.

1. Unite the following terms. $5x - 3y + 4x - 7y$.

Ans. *The x's, when united, equal $+9x$; and the y's, when united, equal $-10y$.* Ans. $9x - 10y$.

2. $2x + 3y + 3x + 4y + 7x + y + x + 9y$. Ans. $13x + 17y$.

 As fast as the pupil adds the co-efficients, he should slightly cross the term, so that he may know when all the terms have been united. Thus:

$$2x + 3y + 3x + 4y + 7x + y + x + 9y = 13x + 17y.$$

3. $4a - 3x + 2a - x + a + 3a - 3x - x$. Ans. $10a - 8x$.

4. $-2y + 7 - x - y - 3x + 11 - 2y$. Ans. $18 - 5y - 4x$.

5. $-2a - x + 3 - a - 3x + 4 - 4a$. Ans. $-7a - 4x + 7$.

§ 36. 2d. When *similar* quantities have *different* signs, select one kind, and add into one sum all the *positive* co-efficients that belong to them. Then add into another sum all the *negative* co-efficients that belong to them. Subtract the less sum from the greater, and prefix the *sign of the greater* to the difference; annex the common letter. Then select another kind, and proceed in the same manner.

EXAMPLES.

Unite the terms in the following.

6. $3a + 4b + 2b - 3c + 3c + a - b - 3b$.

Ans. *The a's, when united, equal $+4a$; the b's, when united, equal $+6b - 4b$, which equals $+2b$. Then, $-3c + 3c$ balance each other, so as to be equal to 0. The answer is $4a + 2b$.*

7. $2a + b + 4a + 4b + 8 + 6 - 3a - 3b - 4$. Ans. $3a + 2b + 10$.

8. $3x + y + z + 3x + 3y - 4x - 2y - z$. Ans. $2x + 2y$.

9. $18 + 8a + 4x + 7x + 5a - 6x - 7a$. Ans. $18 + 6a + 5x$.

10. $4y + 7z - 4z - y + 2z + 2y - z$. Ans. $5y + 4z$.

NOTE.—It sometimes happens that the negative quantity is greater than the positive quantity. In such cases, the difference will have the sign —.

$$11. 7a-3a+4z+a-2z-4a-6a. \text{ Ans. } -5a+2z.$$

$$12. a+b+3a-c+4a-3c+b. \text{ Ans. } 8a+2b-4c.$$

$$13. 5x+5y+3x-2y. \text{ Ans. } 8x+3y.$$

$$14. b+a+3a-5b+4x-a+3a-x. \text{ Ans. } 6a-4b+3x$$

$$15. 4a-5b-10+2y-x+4. \text{ Ans. } 4a-5b+2y-x-6.$$

$$16. 2x+3y-3x+5y-x-z. \text{ Ans. } -2x+8y-z.$$

$$17. x+z+x-z+x+3y. \text{ Ans. } 3x+3y.$$

$$18. 3a-5-4b+6-2a-3+6b. \text{ Ans. } a+2b-2.$$

$$19. 4x+4+5y+6-2+3x-2y. \text{ Ans. } 7x+3y+8.$$

$$20. 4c+3a-b-3a+c-2a-b. \text{ Ans. } 5c-2a-2b.$$

$$21. 2b-c+3a-b+4c+2d-b-5d. = 3a+3c-4d$$

$$22. a-b+c+d+3a+b-2c-4a+10+a. = -a+2b+2c+11d$$

$$23. a+b+3b+x-7+4x-6+3a-y+2. = 4a+4b+4x-y-1$$

$$24. x-4+10-7x+a-b-10+2a-1. = -6x+3a-b-4$$

$$25. 3a-b+4c+2x-c+4a+5c+7b. = 7a+3b+4c+2x$$

$$26. 10-a+6-b-x+7+3b+2a-40. = -a-b-x-23$$

$$27. 8a-16-z+91+2y-87-3z+14. = 8a+2y-4z+2$$

$$28. 29+46+27+y-32-43+y. = 27+y$$

$$29. 73+4x-36-3x-41+7x-y+2x. = 10x-y-4$$

$$30. a-b-a-b-67-42+7a+3b. = 5a-2b-109$$

$$31. 4x-3y+2a-7x+6y-7a+5x-7y. = -2a+2x-4y$$

$$32. 4x-3x+4+x-2x-5+1+3x-5x. = -2x$$

$$33. 2x+y+9-x-y-9+3x+10xy. = 4x+10xy$$

$$34. 5a+3b-4c+2a-5b+6c+a-4b-2c. = 8a-b+4c$$

$$35. 3a+x+10-5a+2x-15-4a-10x+21. = -6a-x+16$$

$$36. 5a-3b+4a-7b+7a+3b-5a-9b. = 11a-16b$$

$$37. 6a+2x-10+2a-3x+10-2y-3a+2x. = 5a-x-2y$$

III. ADDITION OF COMPOUND QUANTITIES.

§ 37. When two or more expressions that consist of several terms, are to be added together, the operation is represented by writing them one after another; taking care, in all except the first, to insert the signs that are understood. Thus, $a-x$ is added to $y+7$ in the following manner:

$$a-x+y+7, \text{ or } y+7+a-x.$$

For to $a-x$ we add first y , and then 7; or to $y+7$ we add a , and then because we ought to have added $a-x$, we see that we have added x too much, and therefore subtract it.

§ 38. The above example shows that it is of no consequence in what order we write the terms. Their place may be changed at pleasure, provided their signs be preserved.

EXAMPLES.

1. Add the following compound quantities. $2a-8x$, $x-3a$, $-4a-2x$, $4x-a$. Ans. $-6a-5x$.

2. Add $2-x+4y$, $3+3x-y$, $-30-x-2y$, and $1-2x+3y-10z$ together. Ans. $-24-x+4y-10z$.

3. Add $3x+5y-6z$, $-2x-8y-9z$, $20x+2y-3z$, and $x-y+z-4$ together. Ans. $22x-2y-17z-4$.

4. Add $3-2y+z$, $4y-2z+5$, $2-z-y$, and $2z-y-10$. Ans. Nothing.

5. Add $3x-6$, $4x+2$, $6+3x$, $7-2x$, $x+1$. $9x+10$

6. Add $4x-a$, $3a-x$, $4x+a$, $7a-3x$. $10a+x$

7. Add $2+2z+3z$, $3z+8$, $5-2z$, $4z$. $15+10z$

8. Add $4-x$, $-x-3$, $6+2x$, $x+2x-1$. $6+3x$

9. Add $x-21$, $3+4x+x$, $2x+3x+36$. $11x+18$

10. Add x , $x-3$, $x+x-3$, $x+2x-6$. $6x-12$

11. Add $2x-3z-4$, $4z-3x$, $5x-5$, $3-10x+3$. $-6x+z-3$

12. Add $3y-4$, $6-3x$, $7x-5y$, $x-y-2$, $3y+5x-3$. $0-3+1$

Questions.—How are compound quantities added? Why is the sign — preserved in addition? Which expression must be put first?

IV. MULTIPLICATION AND DIVISION OF SIMPLE QUANTITIES.

§ 39. Were we to add a to itself four times, we should write the sum thus, $a+a+a+a$; which, when united, becomes $4a$. Whence we have the

RULE.—*Any literal quantity is multiplied by a number, by putting that number before it as a co-efficient; as, 7 times x is $7x$. 6 times a is $6a$.*

§ 40. If the quantity to be multiplied has already a co-efficient, that co-efficient *only* is to be multiplied. Thus, $3x$ taken four times is $3x+3x+3x+3x$, which when united $=12x$. The co-efficient is multiplied by 4. Thus, 4 times $3x = 12x$.

§ 41. It is evident that if 2 times $3x$ is $6x$, then one-half of $6x$ is $3x$. Whence we learn that a quantity with a numeral co-efficient, *may be divided, by merely dividing that co-efficient*. Thus, $8x$ divided by 2 $=4x$. $12x$ divided by 4 $=3x$, &c.

§ 42. In 1661, Rev. William Oughtred of England, published a work, in which he introduced the sign \times to represent multiplication. Thus, $4 \times 3 = 12$, is read 4 *multiplied by* 3 equals 12; or, 4 *into* 3 *equals* 12.

§ 43. In 1668, Mr. Brancker invented the sign \div for division. The sign is always put *before the divisor*; as, $20 \div 4 = 5$; read 20 divided by 4 equals 5; or, 20 *by* 4, *equals* 5.

But in algebra, division is frequently performed by writing the divisor under the dividend, so as to make a fraction; thus, 20 divided by 4, is $\frac{20}{4}$. See page 48.

Questions.—How do we multiply a literal quantity? Multiply x by 3, 4, 5, 6, &c. to 20. How do we multiply a quantity that has already a co-efficient? Multiply $2x$ by 3, 4, 5, &c. to 12. Multiply $3x$ by the same numbers. Multiply $4x$ by the same. How do we divide a quantity that has a numeral co-efficient? Describe the sign for multiplication. Describe the sign for division.

EQUATIONS.—SECTION 1.

Equations which are solved by merely uniting terms.

In each of the following equations, the object is to find the value of x .

Example 1. $x + 2x = 45 - 15$.

Uniting terms, $3x = 30$.

Now, as we have found that *three* x 's = 30, it is evident that *one* x will be one-third of 30.

Therefore, dividing by 3, $x = 10$.

2. Find x in $8x - 4x - x = 7 + 26 + 51 - 15$

Uniting terms, $3x = 69$

Dividing by 3, $x = 23$.

3. Find x in $10x - 5x + 4x = 56 + 75 + 32 - 1$

Uniting terms, $9x = 162$

Dividing by 9, $x = 18$.

4. Find x in $x + 2x + 3x + 4x = 12 + 35 + 74 - 11$

Uniting terms, $10x = 110$

Dividing by 10, $x = 11$.

5. $8x - 3x + 2x = 46 + 54 + 37 - 4$. Ans. $x = 19$.

6. $4x - 3x + 4x = 29 - 36 + 48 + 14$. Ans. $x = 11$.

7. $6x - 8x + 14x = 12 + 36 + 14 + 22$. Ans. $x = 7$.

8. $5x + 4x + 3x = 49 + 14 + 22 + 11$. Ans. $x = 8$.

9. $7x + x = 14 - 22 - 11 + 41 - 6$. Ans. $x = 2$.

10. $4x - 2x = 96 - 7 + 8 - 15 - 10$. Ans. $x = 36$.

11. $5x - x = 2 + 3 - 15 - 10 + 72$. Ans. $x = 13$.

12. $6x = 7 + 4 + 72 - 51 - 16 - 10$. Ans. $x = 1$.

Questions. What is the most useful application of algebra? § 22. What is an equation? What is the part on the left of = called? What the part on the right? What if there are more terms on one side than on the other? What is a numerical equation? How do we find the value of the unknown quantity? What is this operation called?

$$13. 8x - 7x + 5x - 4x + 3x = 27 - 12. \quad \text{Ans. } x = 3.$$

$$14. 5x - 4x + 2x - 3x + x = 39 - 13. \quad \text{Ans. } x = 26$$

$$15. 17x - 4x - 2x - 5x - x = 57 - 32. \quad \text{Ans. } x = 5.$$

$$16. 14x - 35x + 29x + 47x + x = 504. \quad \text{Ans. } x = 9.$$

EQUATIONS FORMED FROM ARITHMETICAL QUESTIONS.

§ 44. An algebraic problem is a proposition which supposes that an unknown quantity has certain relations with other quantities that are known; and requires the discovery of some arithmetical operation by which the unknown quantity may be ascertained.

§ 45. Problems properly belong to literal algebra; but because questions in numeral algebra are solved in a similar manner, they also are called problems in this work.

§ 46. In the solution of problems, the first thing to be done, is to represent the required answer by x , y , or some other final letters of the alphabet. Then, make in algebraical language, with its explanation, a statement of each of the conditions in the question, in the same manner as if the letter were the true answer, and you were required to prove it.

§ 47. When the question has been fairly stated, it will be found that *two different algebraical expressions* have the *same explanation*. These two expressions must be put in the same line, with the sign $=$ between them, so as to form an *equation*. And then, by reducing the equation, the required result will be found. See § 25 and 26.

1. The sum of \$660 was subscribed for a certain purpose, by two persons, A and B; of which, B gave twice as much as A. What did each of them subscribe?

Questions. What is an algebraic problem? What is the first step in the solution of problems? How must you continue the statement? How is it known when the statement is perfect? What is the second part of the solution?


Stating the question, x dollars = what A gave.
 $2x$ dollars = what B gave.
 $x + 2x$ dollars = what both gave.
 And also, 660 dollars = what both gave.
 Therefore, putting the question into an equation,
 $x + 2x = 660$
 Uniting terms, $3x = 660$
 Dividing by 3, $x = 220$ A's share.
 $2x = 440$ B's share.
 $\underline{660}$ proof.

2. Three persons in partnership, put into the stock \$4800; of which, A put in a certain sum, B twice as much, and C as much as A and B both. What did each man put in?

Stating the question, $x = \text{A's share.}$
 $2x = \text{B's share.}$
 $x + 2x = \text{C's share.}$
 Adding all three shares, $x + 2x + x + 2x = \text{the whole.}$
 $4800 = \text{the whole.}$

Therefore, forming the equation,

$x + 2x + x + 2x = 4800$
 Uniting terms, $6x = 4800$
 Dividing by 6, $x = 800$ A's share.
 $2x = 1600$ B's share.
 $800 + 1600 = 2400$ C's share.
 $\underline{4800}$ proof.

 In all the succeeding problems, the learner should prove his answers.

3. A person told his friend that he gave 108 dollars for his horse and saddle; and that the horse cost 8 times as much as the saddle. What was the cost of each?

Stating the question, $x = \text{price of the saddle.}$
 $8x = \text{" horse.}$
 $x + 8x = \text{" both.}$
 $108 = \text{" do.}$

Forming the equation $x + 8x = 108$

Uniting terms,

$$9x = 108$$

Dividing by 9,

$$x = 12 = \text{price of saddle.}$$

$$96 = \text{ " horse.}$$

It is advisable for the pupil, while performing his sums, to write them on his slate in a manner similar to the three questions above; beginning the statement by making x the answer to the *question*, and throughout the operation keeping *equals under equals*. *And in recitation, the whole of it is to be recited.*

4. A father once said, that his age was six times that of his son; and that both of their ages put together, would amount to 49 years. What was the age of each?

Ans. Son's age 7 years; father's 42.

5. A farmer said that he had four times as many cows as horses, and five times as many sheep as cows; and that the number of them all was 100. How many had he of each sort?

Ans. 4 horses; 16 cows; and 80 sheep.

6. A boy told his sister that he had ten times as many chestnuts as apples, and six times as many walnuts as chestnuts. How many had he of each sort, supposing there were 639 in all? Ans. 9 apples; 90 chestnuts; and 540 walnuts.

7. A school-girl said that she had 120 pins and needles; and that she had seven times as many pins as needles. How many had she of each sort? Ans. 15 needles, and 105 pins.

8. A teacher said that her school consisted of 64 scholars; and that there were three times as many in arithmetic as in algebra, and four times as many in grammar as in arithmetic. How many were there in each study?

Ans. 4 in algebra; 12 in arithmetic; and 48 in grammar.

9. A teacher had four arithmeticians who performed 80 sums in a day. The second did as many as the first, the third twice as many, and the fourth as much as all the other three. How many did each perform? Ans. The first and second, each 10; the third, 20; and the fourth, 40.

10. A person said that he was \$450 in debt. That he owed A a certain sum, B twice as much, and C twice as much as to A and B. How much did he owe each?

Ans. To A \$50, to B \$100, and to C \$300.

11. A person said that he was owing to A a certain sum; to B four times as much; and to C eight times as much; and to D six times as much; so that \$570 would make him even with the world. What was his debt to A? Ans. \$30.


12. A boy bought some oranges and some lemons for 54 cents. The price of the oranges was twice the price of the lemons. How much money did he spend for each sort?

Ans. 18 cents for lemons; and 36 cents for oranges.

13. A boy bought some apples, some pears, and some peaches, an equal number of each sort, for 72 cents. The price of a pear was twice that of an apple, and the price of a peach was 3 times that of an apple. How much money did he give for each kind?

Ans. 12 cents for apples; 24 for pears; 36 for peaches.

14. A man bought 3 sheep and 2 cows for \$60. For each cow, he gave 6 times as much as for a sheep. How much did he give for each?


 If x = price of a sheep, all the sheep will cost three times as much, or $3x$. In the same manner both cows will cost twice as much as one cow. One cow will cost $6x$, and 2 cows will cost $12x$.

Ans. \$4, price of a sheep; and \$24 price of a cow.

15. A gentleman hired 3 men and 2 boys. He gave five times as much to a man as he gave to a boy; and for all of them he gave \$6.80. What was the wages of each?

Ans. A boy's wages was 40 cents, and a man's wages, \$2

16. Two men, who are 560 miles apart, start to meet each other. One goes 30, and the other goes 40 miles a day. In how many days will they meet?

 Each will travel x days. The first will go x times

30 miles, and the second will go x times 40 miles; and both together will go the whole distance. It is also evident that x times 30 is the same as 30 times x ; &c. Ans. 8 days.

17. A farmer hired three labourers for \$50.00; giving to the first \$2.00 a day, to the second \$1.50, and to the third \$1.00. The second worked three times as many days as the first; and the third twice as many days as the second. How many days did each work?

Ans. The first, 4; second, 12; and third, 24 days.

18. A gentleman bought some tea, coffee, and sugar, for \$7.04; giving twice as much a pound for coffee as for sugar, and five times as much for tea as for coffee; and there were 20 pounds of sugar, 12 pounds of coffee, and 2 pounds of tea. What was the price of each? Ans. 11 cents for sugar; 22 cents for coffee; and 110 cents for tea.

19. A bookseller sold to a teacher at one time 10 books, and afterwards 15 more at the same rate. Now the difference between the whole sum received at the latter time, and the whole sum received at the former time, was 60 cents. What was the price of each book? Ans. 12 cents.

20. In fencing a side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards per day, and the other 6 yards per day. How many days did they work to make the whole fence? Ans. 30 days.

SECTION II.

TRANSPOSITION.

TRANSPOSITION BY SUBTRACTION.

§ 48. It often happens that in the first member of the equation, some number has been added to the x 's in order to make them equal to the last member. Thus, in the equation

$$x + 16 = 46$$

we see that 16 has been added to x , to make it equal to 46.

§ 49. Now if x with 16 added, is equal to 46; then x alone must be 16 less than 46; that is, $46 - 16$. So, that if we find, that $x + 16 = 46$, we may know that $x = 46 - 16$; or what is the same, $x = 30$.

§ 50. But this may be proved another way. It is very plain that if we subtract a quantity from one member of an equation, and then subtract the same quantity from the other member of the equation; it will still be the fact that the two members are equal to one another. Thus, a half-dollar = 50 cents. Subtract 2 cents from each member. Then a half-dollar — 2 cents = 50 cents — 2 cents; for each of them is equal to 48 cents.

§ 51. Thus, with the equation that we had above,

$$x + 16 = 46$$

Subtracting 16 from both members, $x + 16 - 16 = 46 - 16$.

Now, in the first member of the equation, we have $+ 16 - 16$, which is of no value at all, for $+ 16$ and $- 16$ balance each other, as has been seen in Ex. 6 under § 36.

Therefore the equation is reduced to $x = 46 - 16$


Uniting terms in the last member, $x = 30$.

Questions. Is the first member always without numeral quantities? When x with another number added, equals a certain number what is x itself equal to? How can this be proved?

EQUATIONS.—SECTION 2.

1. Suppose $x+8+3x=56$; what is the value of x ?
 Uniting terms, $4x+8=56$
 Subtracting 8 from both sides, $4x+8-8=56-8$
 Which is the same as $4x=56-8$
 Uniting terms in the last member, $4x=48$
 Dividing by 4, $x=12$.
2. Suppose $2x+14-x-7=41+2-8$; to find x .
 Uniting terms, $x+7=35$
 Subtracting 7 from both sides, $\left. \begin{array}{l} x+7-7=35-7 \text{ or} \\ x=35-7 \end{array} \right\}$
 Uniting terms, $x=28$.
3. Given $x+5-2x-3+4x=26$, to find x .
 Uniting terms, $3x+2=26$
 Subtracting 2 from both sides, $3x=24$
 Dividing by 3, $x=8$.
4. Given $5x+22-2x=31$, to find x . Ans. $x=3$.
5. Given $4x+20-6=34$, to find x . Ans. $x=5$.
6. Given $3x+12+7x=102$, to find x . Ans. $x=9$.
7. Given $10x-6x+14=62$, to find x . Ans. $x=12$.
8. If $7x-14+5x+20=246$, then $x=20$. For, &c.
9. If $8x+17-5x+3=100+10$, then $x=30$.
10. If $7x-14+3x+35=450-29$, then $x=40$.

PROBLEMS.

 For putting questions into equations, see page 25.

1. What number is that, which, with 5 added to it, will be equal to 40?

Stating the question,

x = the number

$x+5$, = after adding

Forming the equation,

$x+5=40$

Subtracting 5, from both,

$x=35$.

2. A man being asked how many shillings he had, answered, Add 15 to their number, and then subtract 1, and the remainder will be 64. How many shillings had he?

Stating the question, $x =$ number of shillings.

$x + 15 =$ after adding.

$x + 15 - 1 =$ after subtracting.

64 = remainder.

Forming the equation, $x + 15 - 1 = 64$

Uniting terms, $x + 14 = 64$

Subtracting 14 from both sides, $x = 50$.

3. What number is that, which with 9 added to it, will equal 23? Ans. 14.

4. Divide 17 dollars between two persons, so that one may have 4 dollars more than the other.

Stating the question, $x =$ the less share.

$x + 4 =$ the greater.

$x + x + 4 =$ both shares.

17 = both shares.


Forming the equation, $x + x + 4 = 17$

Uniting terms, $2x + 4 = 17$

Subtracting 4 from both sides, $2x = 13$

Dividing by 2, $x = 6\frac{1}{2}$ share of one.

$17 - 6\frac{1}{2} = 10\frac{1}{2}$ " the other.

5. The sum of the ages of a certain man and his wife is 55 years; and his age exceeds hers by 7 years. What is the age of each?  Let $x =$ age of the wife.

Ans. 24 the wife's; 31 the man's.

6. A is 5 years older than B, and B is 4 years older than C; and the sum of their ages is 73 years. What is the age of each?

Stating the question, $x =$ C's age.

$x + 4 =$ B's age.

$x + 4 + 5 =$ A's age.

$x + x + 4 + x + 4 + 5 =$ sum of all of them.

Forming the equation, $x + x + 4 + x + 4 + 5 = 73$.

Ans. C 20 years, B 24, A 29.

7. Two persons were candidates for a certain office, where there were 329 voters. The successful candidate gained his election by a majority of 53. How many voted for each?

Ans. 191 for one, and 138 for the other.

8. A, B, and C, would divide \$200 among themselves, so that B may have \$6 more than A; and C \$8 more than B. How much must each have?

Ans. A must have \$60, B \$66, and C \$74.

9. Divide \$1000 between A, B, and C; so that A shall have \$72 more than B, and C \$100 more than A.

Ans. Give B \$252, A \$324, and C \$424.

10. At a certain election 1296 persons voted, and the successful candidate had a majority of 120. How many voted for each?

Ans. 588 for one, and 708 for the other.

11. A father, who has three sons, leaves them \$8000, specifying in his will that the second shall have \$500 more than the youngest, and that the eldest shall have \$1000 more than the second. What is the share of each? Ans. The eldest had \$3500, the second \$2500, the youngest \$2000.

12. A bin, which held 74 bushels, was filled with a mixture of corn, rye, and oats. In it there were 15 bushels of rye more than of corn; and as much oats as both corn and rye. What was the quantity of each?

Ans. 11 bushels of corn, 26 of rye, and 37 of oats.

13. A draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each? Ans. The first, 35 yards; the second, 50 yards; the third, 74 yards.

SECTION III.

TRANSPOSITION BY ADDITION.

§ 52. It is frequently found that some quantity has been *subtracted* from the x 's; as in the equation $5x-44=76$.

§ 53. Now if we had a dollar minus 5 cents, and wished to make the sum just a dollar, we should add the five cents that are lacking. So in the above equation, we have $5x-44$; and as we wish to make it just $5x$, we must add the 44 to it; and $5x-44+44$ is the same as $5x$. But if we add 44 to one member of the equation, we must also add as much to the other member of the equation.

So that adding 44 to both sides, $5x-44+44=76+44$

Or, $5x=76+44$


Uniting terms, $5x=120$

Dividing by 5, $x=24$.

EQUATIONS.—SECTION 3.

1. Given $x+14+3x-27=51$, to find x . Ans. $x=16$.
2. Given $3x-30-2x=46-7$, to find x . Ans. $x=69$.
3. Given $9x-41+6=88-6$, to find x . Ans. $x=13$.
4. Given $20+3x-46=35-4$, to find x . Ans. $x=19$.
5. Given $4x-39-2x=47$, to find x . Ans. $x=43$.
6. Given $7x+27-46=65$, to find x . Ans. $x=12$.
7. Given $14x-55-8x+14=85$, to find x . Ans. $x=21$.

PROBLEMS.

 For putting questions into equations, see page 25.

1. What number is that, from which 8 being subtracted, the remainder is 45?

Stating the question,

x = the number.

$x-8$ = when 8 is subtracted

Forming the equation,


$x-8=45$

Adding 8 to both sides,

$x=53$

2. What number is that, from which 27 being subtracted, the remainder is 41 ? Ans. 68.

3. A person bought two geese for \$1.40; and gave 16 cents more for one than he did for the other. What did each cost him ?

 In this and the four following questions, x must stand for the first mentioned quantity.

Stating the question,

x = the dearest.

$x - 16$ = the cheapest.

$x + x - 16$ = cost of both.

140 = cost of both.

Forming the equation,

$x + x - 16 = 140$

Uniting terms,

$2x - 16 = 140$

Adding 16 to both sides,

$2x = 156$

Dividing by 2,

$x = 78.$

Ans. 78 cents, and 62 cents.

4. Three men, who are engaged in trade, put in \$2600 as follows: A put in a certain sum; B, \$60 less than A; and C, as much as A and B, lacking \$100. What was each man's share ? Ans. A's \$705; B's \$645; C's \$1250.

5. A purse of \$8000 is to be divided among A, B, and C; so that B may receive \$276 less than A, and C \$1112 less than A and B together. What is each man's share ?

Ans. A's \$2416; B's \$2140; C's \$3444.

6. A gentleman gave to two beggars 49 cents; giving to the first 15 cents less than to the second. How many cents did each receive ? Ans. 32 and 17.

7. A man leaves \$16000 to be divided to his widow, son, and daughter, in such a manner that the son is to have \$2000 less than the widow, and the daughter \$1000 less than the son. What is the share of each ?

Ans. Widow, \$7000; son, \$5000; daughter, \$4000.

8. Divide the number 60 into three such parts, that the first may exceed the second by 8, and the third by 16.

Ans. 28; 20; and 12


9. Three men having found a purse of \$160, quarreled about the distribution of it. After the quarrel, it was found that A had got a certain sum, and that B had \$30 more than A, but C \$50 less than A. How much did each obtain?

Ans. A \$60; B \$90; C \$10.

10. Three men, A, B, and C, trade in company, with a stock of \$3130; of which B puts in \$350 more than A, and C \$220 less than A. What was the capital of each?

Ans. A's \$1000; B's \$1350; C's \$780.

11. How can an estate of \$9931 be divided between a widow, son, and daughter, in such a manner that the son shall have \$592 less than the widow, and \$522 more than the daughter?

 After knowing the son's share, how can the daughter's be found?

Ans. Widow, \$3879; son, \$3287; daughter, \$2765.

12. A father divided \$12000 among his three sons, giving to the second \$1500 less than to the eldest, and \$750 more than to the youngest. What was the share of each?

Ans. \$5250; \$3750; and \$3000.

13. A father has willed to his four sons \$25200, as follows: To D a certain sum; to C as much as to D, lacking \$550; to B as much as to C, together with \$1550; and to A twice as much as to B, lacking \$10000. How much does each of them receive?

Ans. A \$5100; B \$7550; C \$6000; D \$6550

SECTION IV.

TRANSPOSITION OF THE UNKNOWN QUANTITY.

§ 54. We have found that when any term has the sign $+$ it may be removed from one member of the equation to the other, if we take care to change the sign to $-$; for this has been done every time we have subtracted a term from both sides. Thus, in the equation

$$x + 5 = 20;$$

if we subtract 5 from both sides, it is plain that the first member becomes x , and the last member becomes $20 - 5$; so that the equation would become

$$x = 20 - 5.$$

§ 55. So also any term that has the sign $-$ may be removed from one member to the other, if we take care to change the sign to $+$. Because this is the same as adding that term to both sides. Thus, in the equation

$$x - 5 = 20,$$

if we add 5 to both sides, the first member becomes x , and the last member becomes $20 + 5$. So that the equation becomes

$$x = 20 + 5.$$

§ 56. When we remove a term from one member of an equation to the other member, we say that we *transpose* that term; and the operation of doing it is called *transposition*.

§ 57. Any term may be transposed from one member of an equation to the other, care being taken *to change the sign when we change the side*.

§ 58. It was stated in § 25, that an equation must be

Questions. How can a positive quantity be removed from one member of an equation to the other? Why? How may a negative quantity be removed from one side to the other? Why? What do we call this method of removing a term? What care is required in transposing? Why are we ever obliged to transpose?

brought so that the unknown quantity will occupy one member of the equation, and the known quantities embrace the other member. And, as it frequently happens that the *unknown* quantities are on both sides, we are obliged to resort to transposition in order to make one side free from them. And likewise, it is often necessary to transpose *known* quantities from the member which contains the unknown quantity.

RULE FOR TRANSPOSING.

§ 59. In transposing, it is generally best to write first the *unknown quantity* that is *already* on the left; and then bring over all those which are on the right, if there are any there. And in transposing the *known quantities* to the right hand member, write those that are *already* there, and then transpose after them what known quantities there are in the left.

EQUATIONS.—SECTION 4.

1. Reduce the equation $4x - 14 = 3x + 12$.

$$\begin{array}{l} \text{Transposing } 3x, \} \\ \text{Transposing } 14, \} \end{array} \quad 4x - 3x = 12 + 14.*$$

$$\text{Uniting terms,} \quad x = 26.$$

2. Given $21 - 7x = 40 - 11x$, to find x . Ans. $x = 4\frac{1}{2}$.

3. Given $40 - 6z = 136 - 14z$, to find z . Ans. $z = 12$.

4. Given $3y - 4 = y + 12$, to find y . Ans. $y = 8$.

5. Given $5x - 15 = 2x + 6$, to find x . Ans. $x = 7$.

6. Given $40 - 6x - 16 = 120 - 14x$, to find x . Ans. 12.

7. Given $4 - 9y = 14 - 11y$, to find y . Ans. $y = 5$.

8. Given $x + 18 = 3x - 5$, to find x .

$$\text{Transposing,} \quad x - 3x = -5 - 18$$

$$\text{Uniting terms,} \quad -2x = -23$$

$$\text{Dividing by 2,} \quad -x = -11\frac{1}{2}$$

Question. In what order do we write the terms when we are transposing?

* The pupil will recite the left hand member of this line, for transposing $3x$.


§ 60. It is of no consequence what sign accompanies the final result; as the *magnitude* of the quantity is not affected by the sign. If we remember that $+$ is understood and may be written with every positive quantity, it will be very evident that the equation $-x = -11\frac{1}{2}$ is just as good as the equation $+x = +11\frac{1}{2}$. In both cases, the quantity x is equal to the number $11\frac{1}{2}$.

§ 61. In the result of the last question, $11\frac{1}{2}$ may be transposed to the first member; and x may be transposed to the last member. Of course, this will change the signs; and the equation will become $11\frac{1}{2} = x$. And if $11\frac{1}{2} = x$, it is evident that $x = 11\frac{1}{2}$. This coincides with what was shown in § 60.

§ 62. From what has just been said, we see that all the terms of each member may be transposed, so that the sign of each term may be changed; and still the equation shall retain the same members as at first, though differently placed. Hence, it is immaterial which member is written first. And also, in any equation *the signs of all the terms may be changed without affecting the equality*.

§ 63. It is evident that all the terms of one member may be transposed to the other member. When this has been done, the member *from which* the terms have been transposed becomes, 0. Thus the equation $x = 11\frac{1}{2}$, may be made $x - 11\frac{1}{2} = 0$; where $-11\frac{1}{2}$ balances x .

PROBLEMS.

 After the equation has been formed by § 47, it must be transposed by § 59.

1. A man has six sons, whose successive ages differ by 4 years; and the eldest is three times as old as the youngest. What are their ages?

Questions. What sign must accompany the answer? Explain. Explain by transposition. To what extent may the signs be changed? Why? To what extent may the terms be transposed? Why?

Stating the question, $x =$ age of the youngest
 $x+4 =$ “ “ next
 $x+4+4 =$ “ “ next
 $x+4+4+4 =$ “ “ next.
 $x+4+4+4+4 =$ “ “ next.
 $x+4+4+4+4+4 =$ “ “ eldest.
 $3x =$ “ “ eldest.

Forming the equation, $x+4+4+4+4+4 = 3x$

Uniting terms, $x+20 = 3x$

Transposing the $3x$ and 20 , $x-3x = -20$

Uniting terms, $-2x = -20$

Dividing by 2 , $-x = -10$

or $x = 10$ age of the
 [youngest.

2. A person bought two horses, and also a hundred dollar harness. The first horse, with the harness, was of equal value with the second horse. But the second horse with the harness cost twice as much as the first. What was the price of each horse?

Stating the question, $x =$ price of the first.
 $x+100 =$ price of the second.
 $x+100+100 = 2$ horse harnessed.
 $2x =$ twice price of first.


Forming the equation, $x+100+100 = 2x$

Transpos. from both members, $x-2x = -100-100$


Uniting terms, $-x = -200$

Or $x = 200 \text{ \&c.}$

3. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, sailing at the rate of 8 miles an hour. How many hours can the ship run before she will be overtaken by the privateer?


 In x hours, the privateer will go $10x$ miles, which is the whole distance. In the same time, the ship will go $8x$ miles. But the ship had already gone 18 miles, which, added to the $8x$, will make the whole distance.

4. A gentleman distributing money among some poor people, found that if he undertook to give 5s. to each, he would lack 10s. Therefore, he gave only 4s. to each, and finds that he has 5s. left. How many persons were there?

 It will be found that $5x - 10 =$ his money by the first supposition, and $4x + 5 =$ the money by the last supposition.

Ans. 15.

5. I once had \$84 in my possession; and I gave away so much of it, that what I have now equals three times as much as I gave away. How much did I give away?

 If I gave away x , then $\$84 - x$ will be what remains.

Ans. \$21.


6. A certain sum of money was shared among five persons, A, B, C, D, and E. Now B received \$10 less than A; C, \$16 more than B; D, \$5 less than C; E, \$15 more than D. And it was found that the sum of the shares of the first three put together, were equal to the sum of the shares of the other two. How much did each man receive?

Ans. A \$21; B \$11; C \$27; D \$22; E \$37.

7. A person wishes to give 3 cents apiece to some beggars, but finds that he has not money enough by 8 cents. He gives them 2 cents apiece and has 3 cents left. How many beggars were there?

Ans. 11.

8. A courier who had started from a certain place 10 hours ago, is pursued by another from the same place, and on the same road. The first goes 4 miles an hour, and the second 9. In how many hours will the pursuer overtake the first?

 If the pursuer goes x hours, the first must go $x + 10$ hours. But, as both go from the same place, the distance that each goes must be the same.

In generalization see page 113.

SECTION V.


MULTIPLICATION OF COMPOUND QUANTITIES
BY SIMPLE QUANTITIES.

§ 64. Suppose you purchase 8 melons at 7 cents apiece, and afterwards find that you must give 5 cents apiece more for them. In this case you pay 8 times 7 cents, and also 8 times 5 cents; that is, first, 56 cents, and afterwards 40 cents.

§ 65. Let us apply this principle to Algebra. You pay in all, 8 times $(7 + 5)$, which $= 56 + 40$. Which shows that *in multiplying a compound quantity, you multiply each term by the multiplier.*

We can easily see that this operation will give the right answer; for in the case of the melons, they cost 12 cents apiece, and therefore their whole cost was 8 times 12 cents which $= 96$ cents. But the answer just obtained, $56 + 40 = 96$.

§ 66. But suppose that after you had paid 7 cents apiece, a deduction of 5 cents apiece was made. The whole cost would then be 8 times $(7 - 5)$, which $= 56 - 40$. And this agrees with the truth; for you first paid 56 cents, and afterwards 40 cents were deducted.

 -5 multiplied by 8, signifies that -5 is to be added 8 times. Therefore retaining the sign, (§ 29), we add -40 .

§ 67. This shows that $+$ multiplied by $+$, produces $+$; and $-$ multiplied by $+$, produces $-$.

EXAMPLES.

1. Multiply $x + 4$, by 3.

$$\begin{array}{r} \text{Operation.} \quad x + 4 \\ \quad \quad \quad 3 \\ \hline \text{Ans.} \quad 3x + 12. \end{array}$$

Questions. How do we multiply a compound quantity? Explain why. How do the signs of the answer correspond with the quantity that is multiplied? Explain for the $-$.

- | | |
|--------------------------------|-----------------|
| 2. Multiply $12+x$, by 5. | Ans. $60+5x$. |
| 3. Multiply $x-10$, by 8. | Ans. $8x-80$. |
| 4. Multiply $126-x$, by 4. | Ans. $504-4x$. |
| 5. Multiply $x+8$, by 6. | |
| 6. Multiply $40+x$, by 10. | |
| 7. Multiply $x-32$, by 9. | |
| 8. Multiply $52-x$, by 12. | |
| 9. Multiply $2x+14$, by 7. | |
| 10. Multiply $27+3x$, by 14. | |
| 11. Multiply $3x-62$, by 15. | |
| 12. Multiply $97-4x$, by 12. | |
| 13. Multiply $x+7-y$, by 7. | |
| 14. Multiply $3x+y-12$, by 8. | |
| 15. Multiply $2x-3y-6$, by 6. | |
| 16. Multiply $3x-12+y$, by 5. | |

§ 68. Franciscus Vieta, a Frenchman, introduced about the year 1600, the *vinculum* or a straight line drawn over the top of two or more quantities when it is wished to connect them together. Thus, $\overline{x+4} \times 3$, signifies that both x and 4 are to be multiplied by 3.

§ 69. In 1629, Albert Girard, a Dutchman, introduced the *parenthesis* as a convenient substitute, in many cases, for the *vinculum*. Thus, $(x+4) \times 3$, is the same as $\overline{x+4} \times 3$; and is read, $x+4$, both $\times 3$. If there are more than two terms under the *vinculum*, we say, after repeating those terms, *all*, &c. Thus, $(x+y) \times (a-b+c)$, is read $x+y$ both into $a-b+c$ *all*. See also § 103.

Questions. What is a *vinculum*? How is the *parenthesis* used? How is a compound quantity read when embraced by a *vinculum* or *parenthesis*?

EQUATIONS.—SECTION 5.

1. Given
- $\overline{x-9} \times 11 = 121$
- , to find
- x
- .

Writing the equation, $\overline{x-9} \times 11 = 121$ Performing the multiplication, $11x-99 = 121$ Transposing and uniting, $11x = 220$ Dividing by 11, $x = 20$.


2. Given
- $(x+7) \times 6 = 54$
- , to find
- x
- . Ans.
- $x = 2$
- .

3. Given
- $\overline{12+x} \times 5 = 100$
- , to find
- x
- . Ans. 8.

4. Given
- $\overline{x-9} \times 8 = 96$
- , to find
- x
- . Ans. 21.

5. Given
- $\overline{367-3x} \times 5 = 920$
- , to find
- x
- . Ans. 61.

6. Given
- $(8+x) \times 2 + 14 = 72$
- , to find
- x
- .

 The pupil must understand that 14 is *not* a part of the multiplier, because there is the sign $+$ *between* it and the multiplier.

Ans. 21.

7. Given
- $(15+x) \times 3 - 27 = 48$
- , to find
- x
- . Ans. 10.

- 8.
- $(112-2x) \times 3 = (2x-7) \times 4$
- , to find
- x
- . Ans. 26.

- 9.
- $(3x+14) \times 4 = (78-x) \times 5$
- , to find
- x
- . Ans.
- $19\frac{1}{4}$
- .

- 10.
- $\overline{2x+8} \times 5 = (32+x) \times 3$
- , to find
- x
- . Ans. 8.

- 11.
- $(3x-14) \times 7 = (17-x) \times 6$
- , to find
- x
- . Ans.
- $7\frac{1}{2}$
- .

- 12.
- $\overline{120-3x} \times 2 = (4x-6) \times 9$
- , to find
- x
- . Ans. 7.

PROBLEMS.

1. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87; and now A's money is double of B's. What did each lay out?

Stating the question,

 x , = the sum for each. $x+126$ = A's sum now. $x-87$ = B's sum now. $2x-174$ = the double of B's

Forming the equation,

 $x+126 = 2x-174$.

Transposing and uniting,

 $-x = -300$.

Changing signs,

 $x = 300$ the answer

2. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old. What were their ages on their wedding day?

Stating the question,

$x =$ the wife's age.

$3x =$ the man's age.

$x + 15 =$ the wife's after 15 years

$3x + 15 =$ the man's after 15 years.

$2x + 30 =$ twice the wife's age.


Forming the equation, $3x + 15 = 2x + 30$

Transposing and uniting,

$x = 15$ the wife's age.

$3x = 45$ the man's age.

3. A man having some calves and some sheep, and being asked how many he had of each sort; answered that he had twenty more sheep than calves, and that seven times the number of calves was equal to three times the number of sheep. How many were there of each?


 If $x =$ number of calves, then $x + 20 =$ number of sheep.

Ans. 15 calves, and 35 sheep.

4. Two persons, A and B, having received equal sums of money, A paid away \$25, and B paid away \$60; and then it appeared that A had just twice as much money left as B. What was the sum that each received?

Ans. \$95.

5. Divide the number 75 into two such parts, so that three times the greater may exceed 7 times the less by 15.

 If $x =$ the greater then $75 - x =$ the less; and $3x$ will $= (7 \text{ times the less}) + 15$.

Ans. 54 and 21.


6. The garrison of a certain town consists of 125 men, partly cavalry and partly infantry. The monthly pay of a horse soldier is \$20, and that of a foot soldier is \$15; and the whole garrison receives \$2050 a month. What is the number of cavalry, and what of infantry?

If $x =$ number of cavalry, then $20x =$ the whole pay of cavalry, &c.

Ans. 35 cavalry, and 90 infantry


7. A grocer sold his brandy for 25 cents a gallon more than he asked for his wine; and 37 gallons of his wine came to as much as 32 gallons of his brandy. What was each per gallon? Ans. \$1.60 for wine; and \$1.85 for brandy.

8. A wine merchant has two kinds of wine; the one costs 9 shillings a gallon, the other 5. He wishes to mix both wines together, so that he may have 50 gallons that may be sold without profit or loss for 8 shillings a gallon. How much must he take of each sort?

 There are 50 gallons of both kinds, and after finding the cost by the kinds, the whole mixture will be worth 50 times 8 shillings.

Ans. $37\frac{1}{2}$ gallons of the best; and $12\frac{1}{2}$ of the poorer.

9. A gentleman is now 40 years old, and his son is 9 years old. In how many years, if they both live, will the father be only twice as old as his son?

 In x years he will be $40+x$, and his son $9+x$.

Ans. 22 years.


10. A man bought 20 oranges and 25 lemons for \$1.95. For each of the oranges he gave 3 cents more than for a lemon. What did he give apiece for each?

Ans. 3 cents for lemons; 6 cents for oranges.

11. A man sold 45 barrels of flour for \$279; some at \$5 a barrel, and some at \$8. How many barrels were there of each sort?

Ans. 27 at \$5; and 18 at \$8.

12. Says John to William, I have three times as many marbles as you. Yes, says William; but if you will give me 20, I shall have 7 times as many as you. How many has each?

 Let x = William's, and $3x$ = John's. Then after the change, $x+20$ = William's, and $3x-20$ = John's.

Ans. John 24; William 8.

13. A person bought a chaise, horse, and harness, for \$440. The horse cost him the price of the harness, and \$80 more; and the chaise cost twice the price of the horse. What did he give for each?

Ans. For the harness \$50; horse \$130; chaise \$260

14. Two men talking of their ages, the first says, Your age is 18 years more than mine, and twice your age is equal to three times mine. What is the age of each?

Ans. Youngest, 36 years; eldest, 54 years.

15. A boy had 41 apples which he wished to divide among three companions as follows: to the second, twice as many as to the first, and 3 apples more; and to the third, three times as many as to the second, and 2 apples more. How many did he give to each?

Ans. To the first, 3; second, 9; third, 29.

+ 16. How many gallons of wine, at 9 shillings a gallon, must be mixed with 20 gallons at 13 shillings, so that the mixture may be worth 10 shillings a gallon? Ans. 60 gallons.

17. Two persons, A and B, have each an annual income of \$400. A spends, every year, \$40 more than B; and, at the end of 4 years, they both together save a sum equal to the yearly income of either. What do they spend annually?

Ans. A, \$370; B, \$330.

18. A farmer wishes to mix rye worth 72 cents a bushel, with oats worth 45 cents a bushel, so that he may have 100 bushels worth 54 cents a bushel. How many bushels of each sort must he take? Ans. $33\frac{1}{3}$ of rye, and $66\frac{2}{3}$ of oats.

In generalization see page 136.

SECTION VI.

FRACTIONS.

§ 70. All the division which the pupil has as yet performed, has been the division either of numeral quantities, or of the numeral co-efficients. But in Algebra, it is frequently necessary to divide *literal* quantities. For example, after having made x to stand for an unknown quantity, we may wish to find the *half* of x , or the *third* of x , or the *fourth* of x , &c.

§ 71. In common arithmetic, if we wish to divide 1 by 2, we do it by writing 2 under the 1; thus, $\frac{1}{2}$. So if we wish to divide 2 by 3, we write 3 under the 2; thus, $\frac{2}{3}$. In the same manner, 2 divided by 5 is written $\frac{2}{5}$; $3 \div 4$ is written $\frac{3}{4}$; $6 \div 7$ is written $\frac{6}{7}$. The quantities that are obtained by dividing in *this manner*, are called *fractions*.

§ 72. In Algebra, we most generally make use of this method of dividing; especially when we divide *literal* quantities. Or, in other words, *we divide a literal quantity by writing the divisor under the dividend, with a straight line between them*; thus, x divided by 2, is written $\frac{x}{2}$; and is read *x-half*; $x \div 3$, is written $\frac{x}{3}$; and is read, *x-third*; $x \div 4$, is written $\frac{x}{4}$, and is read *x-fourth*; $3x \div 4 = \frac{3x}{4}$, and is read, *3x-fourth*.

§ 73. The two separate numbers that we employ in writing a fraction, are called *terms*. The upper term is called the *numerator*, and the lower term is called the *denominator*. Thus, in the fraction $\frac{x}{3}$, we call x the numerator, and 3 the denominator.

§ 74. If the one-third of x is $\frac{x}{3}$, it is evident that $\frac{2}{3}$ of x , is two times as much; that is $\frac{2x}{3}$. If $\frac{1}{5}$ of x is $\frac{x}{5}$, then $\frac{3}{5}$ of

Questions. Can literal quantities be divided? In what manner? Which part of a fraction is the denominator? Which is the numerator? What are they both called?

x is $\frac{3x}{5}$. Whence the rule *to multiply a whole number by a fraction, is, to multiply the whole number by the numerator, and divide by the denominator*; as $\frac{4}{5}$ of x is $\frac{4x}{5}$; $\frac{3}{7}$ of y is $\frac{3y}{7}$; $\frac{2}{3}$ of a is $\frac{2a}{3}$.

Examples. The pupil may multiply a , x , and y , each of them by $\frac{2}{3}$; and then by $\frac{3}{4}$; and then by $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{5}{7}$, $\frac{3}{7}$, successively.

§ 75. As we can multiply a number of *parts* as well as a number of *wholes*, and as the denominator is nothing more than the *name* of the parts; it is plain that *to multiply a fraction, we multiply the numerator, and retain the denominator without alteration*. Thus, 2 times $\frac{3}{5}$ is $\frac{6}{5}$; 3 times $\frac{5}{6}$ is $\frac{15}{6}$; 2 times $\frac{x}{3}$ is $\frac{2x}{3}$; 4 times $\frac{2a}{3}$ is $\frac{8a}{3}$; &c.

Examples. Multiply each of the following fractions by 2, then by 3, and then by 4. $\frac{2}{3}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{6}{7}$, $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$, $\frac{2x}{3}$, $\frac{3x}{4}$, $\frac{2x}{5}$, $\frac{2x}{5}$, $\frac{4x}{7}$.

§ 76. As we know that 2 halves = a whole, we readily conclude that 4 halves = 2 wholes; and that 6 halves = 3 wholes, &c. Likewise, because 3 thirds = a whole, 6 thirds must equal 2 wholes; and 9 thirds must equal 3 wholes. In the same manner 8 fourths = 2 wholes; 20 fifths = four wholes; 18 thirds = 6 wholes, &c. Such fractions as $\frac{8}{4}$, $\frac{20}{5}$, $\frac{18}{3}$, &c., are called *improper fractions*.

§ 77. Hence, in order to find how many *whole ones* there are in any number of halves, we have only to see how many times *two halves* are contained in that number. Thus, in 10 halves there are as many whole ones as there are 2 halves contained in 10 halves; which is 5. In the same manner, in 12 thirds there are as many whole ones as there are 3 thirds contained in 12 thirds; which is 4.

Questions. How do we multiply a whole number by a fraction? Give the answers to the examples. How do we multiply a fraction? *Examples.* What are improper fractions? How can an improper fraction be changed to a whole number?

§ 78. Thus we have the rule, *to change an improper fraction to a whole number, divide the numerator by the denominator.*

When the answer consists of an integer and a fraction, it is called a *mixed number*.

Examples. 1. How many whole ones in $\frac{8}{3}$?

Ans. $8 \div 3 = 2\frac{2}{3}$.

2. How many whole ones in $\frac{7}{2}$? $\frac{12}{5}$? $\frac{14}{3}$? $\frac{20}{4}$? $\frac{25}{5}$?

3. How many whole ones in $\frac{20}{2}$? $\frac{26}{3}$? $\frac{22}{4}$? $\frac{36}{5}$? $\frac{40}{6}$?

4. How many whole x 's in $\frac{6x}{2}$? $\frac{12x}{3}$? $\frac{20x}{4}$? $\frac{27x}{3}$?

5. How many whole x 's in $\frac{10x}{2}$? $\frac{30x}{3}$? $\frac{48x}{4}$? $\frac{50x}{5}$?

6. How many whole x 's in 3 times $\frac{2x}{6}$?

7. How many whole x 's in 4 times $\frac{x}{2}$?

8. How many whole x 's in 5 times $\frac{3x}{5}$? ~~$\frac{3x}{5}$~~

§ 79. If we have the quantity $\frac{x}{5}$, we know that, as it takes 5 fifths to make a whole one, it will take 5 times this quantity to make a whole x . Therefore, if we multiply $\frac{x}{5}$ by 5, we shall obtain $\frac{5x}{5}$, or exactly x . If we multiply $\frac{x}{3}$ by 3, we shall obtain $\frac{3x}{3}$, or, which is the same, x . If we multiply $\frac{x}{4}$ by 4, we shall obtain $\frac{4x}{4}$, or x .

§ 80. As 4 times $\frac{x}{4}$ is equal to x ; then 4 times $\frac{2x}{4}$ must be equal to twice as much, or $2x$; and 4 times $\frac{3x}{4}$ must be three times as much, or $3x$. As 3 times $\frac{x}{3}$ is equal to x ; so 3 times $\frac{2x}{3}$ must be twice as much, or $2x$.

§ 81. *Any fraction when multiplied by the number which is the same as the denominator, will produce a quantity which is the same as the numerator.* Thus,

$$\frac{5x}{4} \times 4 = 5x; \quad \frac{7x}{3} \times 3 = 7x.$$

We shall be able to make use of this principle in the solution of many equations, if we operate in accordance with the following *axiom* or self-evident truth.

Questions. What is a mixed number? What is obtained by multiplying a fraction by a number that is the same as the denominator?

§ 82. *If equals be multiplied by the same, their products will be equal.* Thus, if $x = 10$, then $2x = 20$; $4x = 40$, &c.

§ 83. It is evident [§ 76] that each of the following fractions, $\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \frac{9}{9}$, &c., is equal to 1. Therefore, they must be equal to one another. Also, each of the following fractions, $\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{12}{6}$, &c., is equal to 2; and consequently they are all equal to one another. In the same manner, we may make many fractions that will equal 3; and so of any other number.

Fractions that are equal to one another but have different terms, are called *equivalent fractions*.

§ 84. Let us take from the first set of the above fractions, $\frac{2}{2}$ and $\frac{4}{4}$ which are equal to one another. We see that both the numerator and denominator in the last fraction are twice as much as in the first. We find the same, by taking from the second set, the equal fractions $\frac{2}{1}, \frac{4}{2}$; and also $\frac{4}{2}$ and $\frac{8}{4}$; and also $\frac{6}{3}$ and $\frac{12}{6}$.

Again, in the equal fractions, $\frac{2}{2}$ and $\frac{6}{6}$, we find each term in the last fraction three times as great as the correspondent term in the first fraction. The same may be observed in the fractions $\frac{3}{3}$ and $\frac{9}{9}$; and also in $\frac{2}{1}$ and $\frac{6}{3}$ and also in $\frac{4}{2}$ and $\frac{8}{4}$.

§ 85. By pursuing this investigation, we shall find that *whenever we multiply both the numerator and the denominator by the same number, no matter what that number may be, the fraction made by that multiplication will be equal in value to the first fraction.* Hence, there is an equality between the value of the following fractions, $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}$, &c.; and also between the following, $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \frac{6}{18}$, &c.

§ 86. The principle just explained, leads to another which is of much importance. Suppose we multiply by 8, both terms of the fraction $\frac{2}{3}$; and obtain $\frac{16}{24}$. It is plain that both terms of the fraction $\frac{16}{24}$ can be divided by 8, to bring the frac-

Questions. How can we multiply without destroying the equation? What are equivalent fractions? What will be the effect of multiplying both terms of a fraction by any number?

tion back to $\frac{3}{4}$. So also, if both terms in $\frac{3}{4}$ be multiplied by 6, the fraction will be $\frac{18}{24}$, which means just as much as $\frac{3}{4}$; and, of course, if both terms in $\frac{18}{24}$ be divided by 6, the fraction will be brought back to $\frac{3}{4}$, which is equal to $\frac{18}{24}$. So, in general, *if we divide both the numerator and the denominator of a fraction by the same number, we have a new fraction which will be equal to the first.* Thus, $\frac{8}{10}$ may be changed to $\frac{4}{5}$; $\frac{12}{18}$ to $\frac{2}{3}$; $\frac{15}{21}$ to $\frac{5}{7}$.

§ 87. It is evident, that of several fractions of equal value, that which has the least denominator is the most easily understood. Thus, $\frac{5}{6}$ of an apple is much better known at first sight, than $\frac{35}{42}$ of an apple. And *when a fraction is brought to as small a denominator as it can be changed to, we say it is reduced to its lowest terms.*

§ 88. In order to reduce a fraction to its lowest terms, *divide both the numerator and the denominator by any number that will divide each without a remainder.* Thus, in the fraction $\frac{75}{105}$, both terms may be divided by 5, by which we obtain $\frac{15}{21}$; and both terms of this last fraction may be divided by 3, by which we obtain $\frac{5}{7}$.

EQUATIONS.—SECTION 6.

PROBLEMS.

1. In an orchard, $\frac{1}{4}$ of the trees bear apples, $\frac{1}{5}$ of them bear pears, $\frac{2}{11}$ bear plums, and 81 bear cherries. How many trees are there in the orchard; and how many of each sort?

Stating the question,

x = number of trees.

$\frac{x}{4}$ = apple trees.

$\frac{x}{5}$ = pear trees.

$\frac{2x}{11}$ = plum tree.

81 = cherry trees.

All these trees together = the whole number

Forming the equation, $x = \frac{x}{4} + \frac{x}{5} + \frac{2x}{11} + 81$.

Questions. Suppose we divide both by any number? How do we reduce a fraction to its lowest terms?

Now, we know that if we multiply $\frac{x}{4}$ by 4 we obtain x alone; that is, we *destroy the fraction*, and make it a whole number. And we know, that if we multiply the first member by 4, and also the last member by 4, we shall not destroy the *equation*. See § 82. We will therefore multiply both members by 4, for the purpose of destroying the *fraction* in the first term. It will then become

$$4x = x + \frac{4x}{5} + \frac{8x}{11} + 324.$$

Next, we will multiply *both members* by 5, to destroy the fraction in the second term. This will make

$$20x = 5x + 4x + \frac{40x}{11} + 1620.$$


Then we will multiply by 11, to destroy the remaining *fraction*, which will make

$$220x = 55x + 44x + 40x + 17820.$$

$$\text{Transposing and uniting,} \quad 81x = 17820.$$

$$\text{Dividing by 81,} \quad x = 220. \quad \text{the Ans.}$$

2. In a certain school, $\frac{1}{3}$ of the boys learn mathematics, $\frac{3}{4}$ of them study Latin and Greek, and 6 study English grammar. What is the whole number of scholars?

 After the question is stated, the equation will become


$$x = \frac{x}{3} + \frac{3x}{4} + 6$$

$$\left. \begin{array}{l} \text{Multiplying by 5, to de-} \\ \text{stroy the first fraction,} \end{array} \right\} \quad 5x = x + \frac{15x}{4} + 30$$

$$\text{Multiplying by 4, to destroy, \&c.} \quad 20x = 4x + 15x + 120$$

$$\text{Transposing and uniting,} \quad x = 120. \quad \text{Ans.}$$

3. A gentleman has an estate, $\frac{1}{6}$ of which is woodland, $\frac{2}{3}$ of it pasture, and 105 acres embrace the pleasure grounds, gardens, and orchards. How many acres does it contain?


 It is generally best to multiply by the *greatest* denominator first, as that will sometimes destroy more than one fraction. This is the case in this question; for 6 times $\frac{2x}{3}$ is $\frac{12x}{3} = 4x$. Ans. 630 acres.

4. After paying away $\frac{1}{4}$ and $\frac{1}{3}$ of my money, I find \$22 yet in my purse. How much had I at first? Ans. \$40.

5. A man bought a lot of ground, for which he agreed to pay as follows: $\frac{1}{4}$ of the money on taking possession, $\frac{1}{3}$ of it in 6 months, and \$250 at the end of the year. How much did he pay in all? Ans. \$600.


6. A post is one-fourth of its length in the mud, one-third of it in the water, and 10 feet above the water. What is its whole length? Ans. 24 feet.

7. In a Christmas pudding, $\frac{1}{4}$ is flour, $\frac{1}{5}$ milk, $\frac{1}{6}$ eggs, $\frac{1}{3}$ suet and fruit, and $\frac{3}{4}$ of a pound of spices and other ingredients. What is the weight of the pudding?

 The equation will be $x = \frac{x}{4} + \frac{x}{5} + \frac{x}{6} + \frac{x}{3} + \frac{3}{4}$.

Ans. 15 pounds.

8. A lady being asked what her age was, replied, If you add $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of my age together, the sum will be 18. How old was she?

 After the question has been stated, the equation will be, $\frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 18$. Ans. 24 years.

9. What sum of money is that, whose $\frac{1}{3}$ part, $\frac{1}{4}$ part, and $\frac{1}{5}$ part, added together, amount to 94 dollars? Ans. \$120.


10. A person found upon beginning the study of his profession, that he had passed $\frac{1}{7}$ of his life before he commenced his education, $\frac{1}{3}$ of it under a private teacher, the same time at a public school, and four years at the university. What was his age? Ans. 21 years.

11. How much money have I in my pocket, when the fourth and fifth part of it together, amount to \$2.25? Ans. \$5.

12. The 3d part of my income, said a person, I expend in board and lodging, the 8th part of it in clothes and washing, the 10th part of it in incidental expenses, and yet I save \$318 a year. What was his yearly income? Ans. \$720.


13. A gentleman bequeaths, in his will, the half of his property to his wife, one-sixth part to each of his two sons, the twelfth part to his sister, and the remaining \$600 to his servant. What was the amount of his property? Ans. \$7200.

14. Of a piece of metal, $\frac{1}{3}$ plus 24 ounces is brass, and $\frac{3}{4}$ minus 42 ounces is copper. What is the weight of the piece?


 The equation, when formed, will be

$$x = \frac{x}{3} + 24 + \frac{3x}{4} - 42. \quad \text{Ans. 216 oz.}$$

15. A farmer mixes a quantity of grain, so that 20 bushels less than $\frac{1}{2}$ of it is barley, and 36 bushels more than $\frac{1}{3}$ of it is oats. How many bushels are there in the whole; and how many of each sort?

 In stating the question; $\frac{x}{2} - 20 =$ barley, and $\frac{x}{3} + 36 =$ oats. Ans. 96 bushels in all; 28 of barley, and 68 of oats.


16. A teacher being asked how many scholars he had, replied, If I had as many more, half as many more, and quarter as many more, I should have 88. How many had he?

 In stating the question, he has x ; and as many more is another x ; &c. Ans. 32.

17. In a mixture of copper, tin, and lead; 16 pounds less than $\frac{1}{2}$ was copper, 12 pounds less than $\frac{1}{3}$ was tin, and 4 pounds more than $\frac{1}{4}$ was lead. What was the weight of the whole mixture; and also of each kind?

Ans. 288 lb.; and also 128 lb., 84 lb., and 76 lb

18. What is that number whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part, by 12?

 To find what $\frac{1}{3}$ of it exceeds $\frac{1}{4}$ of it, subtract $\frac{1}{4}$ of it from $\frac{1}{3}$ of it. The remainder is 12. Ans. 144.

19. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ part by 72? Ans. 540.

20. A certain sum of money is to be divided amongst three persons, A, B, and C, as follows: A is to receive \$3000 less than half of it, B \$1000 less than the third of it, and C \$800 more than the fourth of it. What is the sum to be divided; and what does each receive?

Ans. \$38400; and also, \$16200, \$11800, \$10400.

21. A man driving his geese to market, was met by another, who said, Good morrow, master, with your hundred

geese. He replied, I have not a hundred; but if I had as many more, and half as many more, and two geese and a half, I should have a hundred. How many had he? Ans. 39 geese.

22. A shepherd being asked how many sheep he had, replied, If I had as many more, half as many more, and 7 sheep and a half, I should have just 500. How many sheep had he?

Ans. 197 sheep.


23. If the half, third, and fourth parts of my number, be added together, the sum will be one more than my number. Now, what is my number?

Ans. 12.

24. A says to B, Your age is twice and $\frac{3}{5}$ of my age; and the sum of our ages is 54 years. What is the age of each?

Ans. A's, 15 years; B's, 39 years.

25. A young gentleman having received a fortune, spent $\frac{1}{3}$ of it the first year, $\frac{1}{4}$ of it the second, and $\frac{1}{5}$ of it the third, when he had \$2600 left. What was his whole fortune?

 In stating the question, what was spent, = the whole minus \$2600.

Ans. \$12000.

26. A father leaves four sons, who share his property in the following manner: the first takes half, minus \$3000; the second takes a third, minus \$1000; the third takes exactly a fourth; and the fourth son takes a fifth and \$600. What was the whole fortune, and what did each son receive?

Ans. The whole fortune was \$12000; and each son received \$3000.

In generalization, see page 151.

SECTION VII.

FRACTIONS OF COMPOUND QUANTITIES

§ 89. We have found, § 72, that the algebraical method of dividing, is to write the divisor under the dividend, with a straight line between them. It is plain that compound quantities may be divided in this way, as well as simple quantities. Thus, $14+x$ is divided by 3 as follows:

$$\frac{14+x}{3}$$

§ 90. With the same reason, we find the fraction of a compound number, by multiplying it by the numerator, and writing the denominator under the product. Thus,

$$\frac{2}{3} \text{ of } x-5 \text{ is } \frac{2x-10}{3}$$

which is read $2x-10$, both divided by 3.

EXAMPLES.

1. What is $\frac{3}{4}$ of $8+x$?

$$\text{Ans. } \frac{24+3x}{4}.$$

2. What is $\frac{2}{5}$ of $x-27$?

3. What is $\frac{2}{7}$ of $3x-14$?

4. What is $\frac{3}{8}$ of $9+5x$?

5. What is $\frac{4}{7}$ of $7x-19$?

6. What is $\frac{5}{9}$ of $9x-27$?

§ 91. This may be changed into whole numbers by § 78.

$$\text{Thus, } \frac{45x-135}{9} = 5x-15.$$

7. What is $\frac{4}{11}$ of $86-2x$?

8. What is $\frac{3}{7}$ of $14-5x$?

9. What is $\frac{5}{21}$ of $2x-7$?

EQUATIONS.—SECTION 7.

PROBLEMS.

1. A young gentleman being asked his age, said, It is such that if you add 8 years to it, and then divide by 3, the quotient will be 9. How old was he?

Stating the question,

x = his age.

$x+8$ = with 8 added.

$\frac{x+8}{3}$ = quotient.

Forming the equation,

$\frac{x+8}{3} = 9$

Multiplying by 3,

$x+8 = 27$

Transposing and uniting,

$x = 19$ the Ans.

2. A man being asked what he gave for his horse, replied, that if he had given \$12 more, $\frac{3}{4}$ of the sum would be 84. What was the price of the horse?

Stating the question,

x = the price.

$x+12$ = when increased.

$\frac{3x+36}{4} = \frac{3}{4}$ of the sum.

Forming the equation,

$\frac{3x+36}{4} = 84$

Multiplying by 4,

$3x+36 = 336$

Transposing and uniting,


$3x = 300$

Dividing by 3,

$x = 100$ the Ans.

3. What sum of money is that, from which \$5 being subtracted, two-thirds of the remainder shall be \$40? Ans. \$65.

4. It is required to divide a line that is 15 inches long into two such parts, that one of them may be $\frac{2}{3}$ of the other.


 In stating the question, the parts are x , and $15-x$.


Ans. $6\frac{2}{3}$, and $8\frac{1}{3}$.

5. It is required to find a number, such that if 15 be subtracted from it, $\frac{1}{2}$ of the remainder shall be 100? Ans. 140.

6. Divide the number 46 into two parts, so that when one is divided by 7, and the other by 3, the quotients together may amount to 10. Ans. 28 and 18.

7 A person being asked the time of day, answered that the time past from noon was equal to $\frac{2}{3}$ of the time to midnight. What was the hour?

 From noon to midnight is 12 hours; the part of it already past is x . Ans. 20 minutes after 5.

8. Two men talking of their horses, A says to B, My horse is worth \$25 more than yours; and $\frac{3}{4}$ of the value of your horse is equal to $\frac{2}{3}$ of the value of mine. What is the value of each?  The values will be x , and $x+25$.

Ans. A's, \$125; B's, \$100.

9. Two persons have equal sums of money. One having spent \$39, and the other \$93, the last has but half as much as the first. How much had each? Ans. \$147.

10. A man being asked the value of his horse and chaise, answered that the chaise was worth \$50 more than the horse; and that one-half the value of the horse was equal to one-third the value of the chaise. What was the value of each?

Ans. Horse, \$100; chaise, \$150.

11. What number is that, to which if I add 13, and from $\frac{1}{3}$ of the sum subtract 13, the remainder shall be 13?

Ans. 325.

12. A man being asked the value of his horse and saddle, answered that his horse was worth \$114 more than his saddle, and that $\frac{2}{3}$ of the value of his horse was 7 times the value of his saddle. What was the value of each?

Ans. Saddle, \$12; horse, \$126.

13. A legacy of \$1200 was left between A and B, in such a manner, that $\frac{1}{8}$ of A's share was equal to $\frac{1}{7}$ of B's. What sum did each receive? Ans. A, \$640; B, \$560.

14. A person rented a house on a lease of 21 years, and agreed to do the repairs when $\frac{2}{3}$ of that part of the lease which had elapsed, should equal $\frac{8}{9}$ of the part to come. How much time will elapse before he repairs? Ans. 12 years.

15. What number is that, to which if I add 20, and from $\frac{2}{3}$ of this sum subtract 12, the remainder shall be 10? Ans. 13.

16. A person has a lease for 99 years; and being asked how much of it was already expired, he answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. What time had already past? \times Ans. 54 years.

17. Divide \$183 between two men; so that $\frac{4}{7}$ of what the first receives, shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each? Ans. \$63, and \$120.

18. Bought sheep for \$300, calves for \$100, and pigs for \$25; and then laid out $\frac{5}{6}$ of the rest of my money, which was \$15, in getting them home. How much had I at first?

 15 is one member of the equation. Ans. \$443.

19. A gentleman paid four laborers \$136. To the first he paid three times as much as to the second, wanting \$4; to the third one-half as much as to the first, and \$6 more; and to the fourth four times as much as to the third, and \$5 more. How much did he pay to each?

Ans. To the first, \$26; second, \$10; third, \$19; fourth, \$81.

In generalization, see page 151.

SECTION VIII.

DIVISION OF FRACTIONS, AND FRACTIONS OF FRACTIONS.

§ 92. It was shown in § 75, that in multiplying a fraction, we multiply the numerator only, and retain the denominator. On the same principle, *a fraction is divided by dividing its numerator, and retaining its denominator.*

$$\text{Thus, } \frac{6}{7} \div 3 = \frac{2}{7}. \quad \frac{8x}{9} \div 4 = \frac{2x}{9}, \text{ \&c.}$$

§ 93. But, supposing we wish to divide $\frac{2}{7}$ by 3. In this case, we cannot divide the numerator 2 by 3 without a remainder; and therefore we must look for some other principle to assist us. We shall find it in § 85, where it was shown that a fraction may be changed to one with different terms, without altering the value.

§ 94. It is evident, then, that we have only to change the fraction which is to be divided, to some equivalent fraction, whose numerator can be divided by the divisor without a remainder. Thus, $\frac{2}{7}$ can be changed $\frac{6}{21}$, $\frac{12}{42}$, $\frac{18}{63}$, &c.; each of which can be divided by 3, giving for the quotient either $\frac{2}{7}$, or $\frac{4}{21}$, or $\frac{6}{33}$, &c.

§ 95. The *most convenient* equivalent fraction will be obtained by multiplying both terms of the fraction by the number which is to be the divisor. Because it is certain that after the numerator has been multiplied by a number, the product can in return be *divided* by that number.

$$\text{Thus, } \frac{4}{9} \div 5 = \frac{20}{45} \div 5; \text{ and } \frac{20}{45} \div 5 = \frac{4}{45}.$$

Questions. How do we multiply a fraction? How then must we divide a fraction? How can we do if the numerator cannot be divided without a remainder? How can we obtain the most convenient equivalent fraction?

§ 96. But, by examining the example just given, we find the numerator of the answer to be the same as the numerator of the first fraction; for the first numerator has been multiplied and then divided again by the same number. The *denominator only* is changed; and that has been done by multiplying the first denominator by the number that was to be the divisor.

§ 97. Hence the *rule* for dividing a fraction. *Divide its numerator when it can be done without a remainder. But if there would be a remainder; instead of dividing the numerator, multiply the denominator by the divisor for a new denominator; and leave the numerator as it is.*

$$\text{Thus, } \frac{2}{9} \div 5 = \frac{2}{45}; \quad \frac{7}{11} \div 6 = \frac{7}{66}.$$

EXAMPLES.


- | | |
|---------------------------------|-------------------------|
| 1. Divide $\frac{3x}{4}$ by 5. | Ans. $\frac{3x}{20}$. |
| 2. Divide $\frac{2x}{3}$ by 3. | Ans. $\frac{2x}{9}$. |
| 3. Divide $\frac{4x}{7}$ by 2. | Ans. $\frac{2x}{7}$. |
| 4. Divide $\frac{5x}{3}$ by 5. | Ans. $\frac{x}{3}$. |
| 5. Divide $\frac{12x}{5}$ by 3. | Ans. $\frac{4x}{5}$. |
| 6. Divide $\frac{7+x}{3}$ by 2. | Ans. $\frac{7+x}{6}$. |
| 7. Divide $\frac{x-6}{5}$ by 4. | Ans. $\frac{x-6}{20}$. |

Questions. By this method, how is the numerator of the answer obtained? What effect has this method upon the numerator? What then is the rule for dividing a fraction?

$$8. \text{ What is } \frac{1}{4} \text{ of } \frac{3x-47+y}{16} ? \quad \text{Ans. } \frac{3x-47+y}{64}.$$

$$9. \text{ What is } \frac{1}{3} \text{ of } \frac{7x+4}{x-1} ? \quad \text{Ans. } \frac{7x+4}{5x-5}.$$

$$10. \text{ What is } \frac{1}{7} \text{ of } \frac{21-3x}{2x+9} ? \quad \text{Ans. } \frac{21-3x}{14x+63}.$$

 In this example, although we can divide the first term in the numerator, we do not do it because we cannot divide the last term.

$$11. \text{ What is } \frac{1}{6} \text{ of } \frac{48+7y}{3x-2y} ? \quad \text{Ans. } \frac{48+7y}{18x-12y}.$$

$$12. \text{ What is } \frac{1}{9} \text{ of } \frac{27-6y}{31+x-y} ? \quad \text{Ans. } \frac{27-6y}{279+9x-9y}.$$

$$13. \text{ What is } \frac{1}{8} \text{ of } \frac{32x+16}{x-6+y} ? \quad \text{Ans. } \frac{4x+2}{x-6+y}.$$

§ 98. We are now enabled to find a fraction of a fraction by the rule in § 74; which is to multiply by the numerator, and divide by the denominator. *To multiply by the numerator is to multiply the numerators together. And to divide by the denominator, we have just shown, is to multiply the denominators together.*

EXAMPLES.

$$14. \text{ What is } \frac{2}{3} \text{ of } \frac{5x}{6} ? \quad \text{Ans. } \frac{10x}{18} = \frac{5x}{9}.$$

$$15. \text{ What is } \frac{3}{4} \text{ of } \frac{7x}{8} ? \quad \text{Ans. } \frac{21x}{32}.$$

$$16. \text{ What is } \frac{4}{5} \text{ of } \frac{3x}{7} ? \quad \text{Ans. } \frac{12x}{35}.$$

$$17. \text{ What is } \frac{2}{7} \text{ of } \frac{5x-1}{3} ? \quad \text{Ans. } \frac{10x-2}{21}.$$

$$18. \text{ What is } \frac{3}{5} \text{ of } \frac{71+2x}{4} ? \quad \text{Ans. } \frac{213+6x}{20}.$$

Question. How do we find a fraction of a fraction?

$$19. \text{ What is } \frac{4}{3} \text{ of } \frac{6x}{23+y} ? \quad \text{Ans. } \frac{24x}{207+9y} = \frac{8x}{69+3y}.$$

$$20. \text{ What is } \frac{2}{11} \text{ of } \frac{9y}{x+61} ? \quad \text{Ans. } \frac{18y}{11x+671}.$$

$$21. \text{ What is } \frac{2}{15} \text{ of } \frac{4x-20}{41+6x} ? \quad \text{Ans. } \frac{36x-180}{410+60x} = \frac{18x-90}{205+30x}.$$

$$22. \text{ What is } \frac{4}{7} \text{ of } \frac{3y+7x}{2x-11y} ? \quad \text{Ans. } \frac{12y+28x}{14x-77y}.$$

$$23. \text{ What is } \frac{3}{16} \text{ of } \frac{6x+7y-10}{4+3y-2x} ? \quad \text{Ans. } \frac{18x+21y-30}{40+30y-20x}.$$

$$24. \text{ What is } \frac{4}{9} \text{ of } \frac{x-y-6}{6+x-y} ? \quad \text{Ans. } \frac{4x-4y-24}{54+9x-9y}.$$

$$25. \text{ What is } \frac{2}{3} \text{ of } \frac{3x+5y-8}{27-4x} ? \quad \text{Ans. } \frac{6x+10y-16}{135-20x}.$$

$$26. \text{ What is } \frac{4}{7} \text{ of } \frac{3x-18}{2x+32} ?$$

$$27. \text{ What is } \frac{5}{12} \text{ of } \frac{21+9x-3y}{10-5y+20x} ?$$

$$28. \text{ What is } \frac{3}{8} \text{ of } \frac{16x-8+32y}{9-15x+12y} ?$$

EQUATIONS.—SECTION 8.

1. A farmer wishes to mix 116 bushels of provender, consisting of rye, barley, and oats, so that it may contain $\frac{2}{7}$ as much barley as oats, and $\frac{1}{2}$ as much rye as barley. How much of each must there be in the mixture?

Stating the question, $x = \text{oats}$; and $\frac{5x}{7} = \text{barley}$.

Then, $\frac{1}{2}$ of $\frac{5x}{7}$ is $\frac{5x}{14} = \text{rye}$.

Forming the equation, $x + \frac{5x}{7} + \frac{5x}{14} = 116$

Multiplying by 14, $14x + 10x + 5x = 1624$

Uniting terms, $29x = 1624$

Dividing by 29, $x = 56$ the Ans.

2. I paid away a fourth of my money, and then a fifth of the remainder, which was \$72. How much money had I at first?

Stating the question,

x = what I have.

$\frac{x}{4}$ = paid first.

$x - \frac{x}{4}$ = the remainder.

$\frac{x}{5} - \frac{x}{20}$ = paid afterwards.

Forming the equation,

$\frac{x}{5} - \frac{x}{20} = 72$

Multiplying by 20,

$4x - x = 1440$


Uniting terms,

$3x = 1440$

Dividing by 3,

$x = 480$ the Ans.

3. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had \$72 left. How much had I at first?

 In stating the question, the remainder after the first payment was $x - \frac{x}{4}$; and $\frac{1}{5}$ of that is $\frac{x}{5} - \frac{x}{20}$.

Then, $\frac{x}{4} + \frac{x}{5} - \frac{x}{20} + 72 =$ all my money. Ans. \$120.

4. A clerk spends $\frac{2}{3}$ of his salary for his board, and $\frac{2}{3}$ of the remainder in clothes, and yet saves \$150 a year. What is his yearly salary? Ans. \$1350.

5. Of a detachment of soldiers, $\frac{2}{3}$ are on actual duty, $\frac{1}{3}$ of them sick, $\frac{1}{3}$ of the remainder absent on leave, and the rest, which is 380, have deserted. What was the number of men in the detachment? Ans. 2280 men.

6. A young man, who had just received a fortune, spent $\frac{3}{8}$ of it the first year, and $\frac{4}{5}$ of the remainder the next year; when he had \$1420 left. What was his fortune? Ans. \$11360.


7. If from $\frac{1}{3}$ of my height in inches, 12 be subtracted, $\frac{1}{3}$ of the remainder will be 2. What is my height? Ans 5 ft. 6 in.

8. A Christmas cake was mixed as follows: $\frac{1}{4}$ was sugar, $\frac{1}{6}$ butter and fruit, $\frac{1}{12}$ eggs, and 3 pounds more than half of all these was flour. How much did the cake weigh?

Ans. 12 pounds.

9. A, B, and C, own together a field of 36 acres. B has $\frac{1}{3}$


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more than A, and C has $\frac{1}{4}$ more than B. What is each man's share?  $\frac{1}{3}$ more, signifies once and $\frac{1}{3}$ as much.

Ans. A, 9 acres; B, 12; C, 15

10. A gentleman gave to three persons 98 dollars. The second received $\frac{5}{8}$ of the sum given to the first; and the third, $\frac{1}{5}$ of what the second had. What did each receive?

Ans. \$56, \$35, and \$7.

11. A gentleman invested $\frac{3}{8}$ of his property in a canal. When he sold out, he lost $\frac{2}{3}$ of the sum invested, receiving only \$1446. What was the value of his property when he began?  His investment minus his loss, equals \$1446.

Ans. \$11568.

12. A boy spent $\frac{9}{10}$ of his money for fruit; giving $\frac{2}{3}$ of what he spent for oranges, and 21 cents for lemons. How much money had he?

Ans. 70 cents.

13. A grocer put 490 gallons of beer into three casks; of which the second held $1\frac{1}{3}$ times as much as the first, and the third held $\frac{3}{4}$ as much as both the others. What did each hold?

Ans. 120, 160, and 210 gallons.

14. A gentleman leaves \$315 to be divided among four servants in the following manner: B is to receive as much as A, and $\frac{1}{2}$ as much more; C is to receive as much as A and B, and $\frac{1}{3}$ as much more; D is to receive as much as the other three, and $\frac{1}{4}$ as much more. What is the share of each?

Ans. A, \$24; B, \$36; C, \$80; D, \$175

In generalization, see page 155.

SECTION IX.

SUBTRACTION OF COMPOUND QUANTITIES.

§ 99. Suppose we wish to subtract the expression $x+6$ from y . It is evident that we may first subtract x ; which will give us, $y-x$. But we wish to subtract not only x , but 6 also. Well, after we have subtracted x we will subtract 6 also; and then the answer will be, $y-x-6$. Therefore, whenever we wish to subtract a compound quantity whose terms are all positive, we write them after the other quantity with $+$ changed to $-$.

§ 100. Again, suppose we wish to subtract the expression $x-6$ from y ; in which the number to be subtracted has $-$ instead of $+$. As before, we will first subtract x , by which we obtain $y-x$. But the quantity to be subtracted was 6 less than x ; and we have therefore subtracted 6 too much. We will therefore add 6 to our last answer for the true remainder, which will give us $y-x+6$. Here, we have changed the positive x to $-x$, and the -6 , to $+6$.

RULE FOR SUBTRACTING COMPOUND QUANTITIES.

§ 101. *Change all the signs of the expression which is to be subtracted, the sign $+$ to $-$, and the sign $-$ to $+$; and then write the terms after the other quantity.* It is to be recollected that in the quantity *from which* we subtract, the signs are not altered.

§ 102. Although the subtraction is performed the moment the quantities are written according to the above rule, yet, after that operation, it is always expedient to *unite the terms* if possible.

Questions. How do we subtract a compound quantity whose terms are all positive? Explain why. How do we subtract a negative quantity? Explain why. What is the *Rule* for subtracting compound quantities? What should be done after the quantity is subtracted?

EXAMPLES.

1. Subtract
- $7x+6+y$
- from
- $6y-17$
- .

Ans. *Changing the signs of the quantity we subtract, we have, $6y-17-7x-6-y$; which $= 5y-23-7x$.*

2. Subtract
- $4y+3x-10$
- from
- $74-x$
- .

Ans. $74-x-4y-3x+10$; which $= 84-4x-4y$.

3. Subtract
- $6-x-3y$
- from
- $7x+6y$
- .

Ans. $7x+6y-6+x+3y$; which equals $8x+9y-6$.

4. From
- $4x-3y+27$
- , subtract
- $6y-12+x$
- .

Ans. $3x-9y+39$.

5. From
- $6+x-y$
- , subtract
- $13-9y-x$
- . Ans.
- $2x+8y-7$

6. From
- $8x-2+3y$
- , subtract
- $3y+4-3x$
- . Ans.
- $11x-6$
- .

7. From
- $5+4x$
- , subtract
- $2-5x+4y-z$
- .

Ans. $9x-4y+z+3$

8. From
- $6x-8y$
- , subtract
- $-x-y+50$
- . Ans.
- $7x-7y-50$
- .

9. From
- $14-\frac{2x}{3}$
- , subtract
- $3x-12$
- . Ans.
- $26-3x-\frac{2x}{3}$
- .

10. From
- $\frac{8+7x}{3}$
- , subtract
- $5+\frac{3x}{5}$
- . Ans.
- $\frac{8+7x}{3}-5-\frac{3x}{5}$
- .

11. From
- $7-\frac{2y}{3}$
- , subtract
- $\frac{10y-4}{6}$
- . Ans.
- $7-\frac{2y}{3}-\frac{10y-4}{6}$
- .

§ 103. It has been shown, § 68 and § 69, that any compound quantity may be considered and operated upon as a simple quantity, by merely drawing a vinculum above it, or enclosing it in a parenthesis. Whenever that compound quantity is a fraction, the *line between the numerator and denominator serves as a vinculum*. Thus, in the eleventh example, above, $\frac{10y-4}{6}$ is subtracted as a simple quantity, and

Questions. What effect has a vinculum upon a compound quantity? What if the compound quantity is a fraction? If a fraction is to be subtracted, what sign is to be changed?

is read, *minus the fraction*, &c. Therefore the sign of each term in it is not changed; but — is put before the whole fraction.

$$12. \text{ From } \frac{4x-6}{4}, \text{ subtract } \frac{3x+7}{5}. \quad \text{Ans. } \frac{4x-6}{4} - \frac{3x+7}{5}.$$

$$13. \text{ From } \frac{3x+8}{2}, \text{ subtract } \frac{51-x}{3}. \quad \text{Ans. } \frac{3x+8}{2} - \frac{51-x}{3}.$$

$$14. \text{ From } \frac{7-2x}{3}, \text{ subtract } \frac{21x-4}{10} - x. \\ \text{Ans. } \frac{7-2x}{3} - \frac{21x-4}{10} + x.$$

$$15. \text{ From } x+4-2x, \text{ subtract } 3x-6.$$

$$16. \text{ From } 4x-6y, \text{ subtract } -x+y-7.$$

$$17. \text{ From } 8+2x, \text{ subtract } -y+3x.$$

$$18. \text{ From } \frac{3x-7}{2}, \text{ subtract } \frac{2-6x}{3}.$$

$$19. \text{ From } 8x, \text{ subtract } 3x - \frac{7x-6}{5}.$$

$$20. \text{ From } x, \text{ subtract } \frac{2y+6x}{3} - x.$$

$$21. \text{ From } 2x-y, \text{ subtract } \frac{2x-y}{4} + y.$$

$$22. \text{ From } 3+x, \text{ subtract } \frac{3+x}{7} + x.$$

UNITING FRACTIONS OF DIFFERENT DENOMINATORS.

§ 104. By looking at the answers to the last three examples in § 102, and also the next three in § 103, it will appear that we ought to have some rule for uniting their terms. We can easily find one by applying the principle explained in § 85. For we have only to change each of the fractions to an equivalent fraction, so that they will all have one with another the same common denominator.

Thus, the answer to the twelfth example under § 103, is $\frac{4x-6}{4} - \frac{3x+7}{5}$. Now each of these two fractions may be changed to 20ths, by multiplying the first by 5, and the last by 4. They will then become $\frac{20x-30}{20} - \frac{12x+28}{20}$ which = $\frac{(20x-30) - (12x+28)}{20} = \frac{20x-30-12x-28}{20} = \frac{8x-58}{20} = \frac{4x-29}{10}$.

§ 105. Thus we have, the *rule* for uniting fractions of different denominators. *Multiply all the denominators together for a new denominator; and each numerator by all the denominators except its own, for new numerators; remembering that if a compound numerator follows minus —, all the signs in it must be changed the moment one denominator is used for the whole quantity; that is, when the short vinculums are destroyed for the purpose of making the longer one.*

EXAMPLES.

1. Unite the terms in the answer to the 13th sum in § 103.

Operation.

$$\left(\frac{3x+8}{2} - \frac{51-x}{3}\right) = \left(\frac{9x+24}{6} - \frac{102-2x}{6}\right) = \frac{9x+24-102+2x}{6} = \frac{11x-78}{6}.$$

NOTE.—In this operation the first minus has reference to the whole quantity $\left(\frac{51-x}{3}\right)$; the second minus to the whole quantity $\left(\frac{102-2x}{6}\right)$. In this last quantity, 102 has no sign before *itself*, and is therefore positive. Now, when the line

Questions. How can fractional terms be united? Give the rule. What if integers are multiplied with the fractions?

between the numerator and denominator is carried *through the whole quantity*, the vinculum of $102-2x$ is destroyed; and then is the time for changing the signs for subtracting.

§ 106. Whenever there are integers to be united with fractions, *they* may be changed to fractions, by putting the number 1 under them for the denominator. Thus, $6 = \frac{6}{1}$; $x = \frac{x}{1}$.

2. Unite the terms in the quantity $7 - \frac{2y}{5} - \frac{10y-4}{6}$.

Operation.—The quantity is $\frac{7}{1} - \frac{2y}{5} - \frac{10y-4}{6}$.

Both terms of $\frac{7}{1}$ multiplied by 5 and by 6, equals $\frac{210}{30}$.

Both terms of $-\frac{2y}{5}$ multiplied by 6, equals $-\frac{12y}{30}$.

Both terms of $-\frac{10y-4}{6}$ multiplied by 5, equals $-\frac{50y-20}{30}$.

Therefore, $\left(\frac{7}{1} - \frac{2y}{5} - \frac{10y-4}{6}\right) = \left(\frac{210}{30} - \frac{12y}{30} - \frac{50y-20}{30}\right)$

which $= \frac{210-12y-50y+20}{30} = \frac{230-62y}{30} = \frac{115-31y}{15}$.

3. Unite the terms in the quantity, $\frac{8+7x}{3} - 5 - \frac{3x}{5}$.

Ans. $\frac{40+35x-75-9x}{15} = \frac{26x-35}{15}$

4. Unite the terms in the quantity, $26 - 3x + \frac{2x}{3}$.

Ans. $26 - \frac{9x+2x}{3} = 26 - \frac{7x}{3}$.


5. Unite the terms in the quantity, $7 - \frac{2x}{3} - \frac{21x-4}{10} + x$.

Ans. $\frac{222-53x}{30}$

6. Unite the terms in the quantity, $12 - \frac{6+2x}{4} - \frac{4x-3}{5} - \frac{12x}{8}$.
 Ans. $\frac{11-28x}{10}$.

7. Unite the terms in the quantity, $\frac{x}{2} + \frac{x}{3} + \frac{x}{4}$.
 Ans. $\frac{13x}{12} = x + \frac{x}{12}$.

8. Unite the terms in the quantity, $4 + \frac{x}{3} - 3 - \frac{2x}{9}$.

 The fractions may be united }
 by themselves. } Ans. $1 + \frac{x}{9}$.

9. Unite the terms in the quantity, $4z - \frac{3z}{7} + z - \frac{4z}{5}$.
 Ans. $5z - \frac{43z}{35}$.

10. Unite the terms in the quantity, $y - \frac{3y-4}{5} + 2y - \frac{3+2y}{3}$.
 Ans. $\frac{26y-3}{15} = y + \frac{11y-3}{15}$.

11. Subtract $\frac{6x+7}{4}$ from $\frac{3x-4}{7}$.

12. Subtract $\frac{5x-6}{3}$ from $\frac{8-x}{5}$.

13. Subtract $3 - \frac{x}{7}$ from $7\frac{x}{3}$.

14. Subtract $3x - \frac{2-x}{4}$ from $5x$.

15. Subtract $\frac{5y-3x+7}{5}$ from $\frac{6x}{7} - \frac{3y}{4}$.

16. Subtract $\frac{10x}{3} - \frac{3y+4}{5} + \frac{7x-y}{4}$ from $4x - \frac{y}{3}$.

EQUATIONS.—SECTION 9.

1. There are two numbers, whose sum is 140; and if 4 times the less be subtracted from 3 times the greater, the remainder will be 70. What are the numbers?

Stating the question,

x = the greater.

$140 - x$ = the less.

$3x$ = 3 times the greater.

$560 - 4x$ = 4 times the less.

Forming the equation, $3x - 560 + 4x = 70$

Transposing, uniting, and dividing, $x = 90$

Ans. Greater number, 90; less, 50.


2. A person, after spending \$100 more than a third of his yearly income, found that the remainder was \$150 more than half of it. What was his income?

Ans. \$1500

3. Two men, A and B, commenced trade. A had twice as much money as B; he has since gained \$50, and B has lost \$90; and now the difference between A's and B's money, is equal to three times what B has. How much had each when they commenced trade?

Ans. A, \$410; B, \$205.

4. A man bought a horse and chaise for \$341. If $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from three times the price of the horse. What was the price of each?

 If the price of the chaise be x , and the price of the horse be $341 - x$; then the first remainder will be $2x - \frac{1023 - 3x}{8}$.

But when the fraction is destroyed, the vinculum is taken away, and therefore the last sign must be changed from — to +.

Ans. Chaise, \$189; horse, \$152.

5. A gentleman bought a watch and chain for \$160. If $\frac{3}{4}$ of the price of the watch be subtracted from six times the price of the chain, the remainder will be the same as if $\frac{5}{12}$ of the

price of the chain were subtracted from twice the price of the watch. What was the price of each?

Ans. Watch, \$112; chain, \$48.

6. Divide the number 204 into two such parts, that if $\frac{1}{3}$ of the less were subtracted from the greater, the remainder will be equal to $\frac{2}{3}$ of the greater subtracted from four times the less.

Ans. Greater, 154; less, 50.


7. Two travellers, A and B, found a purse of money. A first takes out \$2 and $\frac{1}{6}$ of what remains; and then B takes \$3 and $\frac{1}{6}$ of what remains; and it is found that each has the same sum. How much money was in the purse? Ans. \$20.

8. A shepherd was met by a band of robbers, who plundered him of half of his flock and half a sheep over. Afterwards a second party met him, and took half of what he had left, and half a sheep over; and soon after this, a third party met him, and treated him in the like manner; and then he had 5 sheep left. How many had he at first? Ans. 47 sheep.

9. A gentleman hired a laborer for 20 days, on condition that for every day he worked he should receive 14 shillings; but for every day he was idle he should forfeit 6 shillings. At the end of the 20 days he received 160 shillings. How many days did he work, and how many days was he idle?

Ans. He worked 14 days, and was idle 6.

10. Divide the number 48 into two such parts, that the excess of one of them above 20, shall be three times as much as the other wants of 20.

 The excess of a number above 20 is obtained by subtracting 20 from it.

Ans. 32 and 16.

11. A person in play lost a fourth of his money, and then won back 3s.; after which he lost a third of what he now had, and then won back 2s.; lastly, he lost a seventh of what he then had, and then found he had but 12s. remaining. What had he at first?

Ans. 20s.

In generalization, see page 158.

SECTION X.

RATIO AND PROPORTION.

§ 107. When an unknown quantity is not, either by itself, or in some connexion with others, known to be *equal* to some known quantity or set of quantities; we may sometimes find that there is a *comparison* between it and some known quantity, which is the same as the comparison between two *known* quantities.

Thus, suppose I buy 27 yards of cloth for \$72, and wish to sell for \$16 so much of it as cost me \$16. In this case the number of yards to be sold is not equal to any other quantity that is mentioned. But we suppose that it must compare with the number of yards bought, in the same manner that \$16 compares with \$72. By knowing this comparison, we can find the number of yards; because, as \$16 is $\frac{2}{9}$ of \$72, so the number of yards to be sold must be $\frac{2}{9}$ of the number of yards bought. It is 6 yards.

§ 108. It will be seen that the comparison in this example consists in observing how many times one of the numbers is contained in the other. 72 is contained in 16, two-ninths of a time. When a comparison of this kind is made, the result that is obtained is called their *RATIO*. Thus, in comparing the numbers 3 and 4, we find that 4 is contained in 3, three-fourths of a time; and therefore we say *the ratio of 3 to 4 is $\frac{3}{4}$* .

§ 109. The pupil must remember that the ratio of one number to another, always signifies how the *first* number compares with the *last*. Thus, the ratio of 8 to 5, is $\frac{8}{5}$; that is, 8 is $\frac{8}{5}$ of 5. Hence the ratio is expressed by making the *first* term to be a *numerator*, and the *last* to be the *denominator*.

Questions. Do we ever make use of *comparison* in algebra? Explain how. In what does the comparison consist? What is that kind of comparison called in mathematical language? How can a ratio be expressed? Why?

§ 110. In the example just furnished relative to the cloth; the ratio of the money paid, to the money obtained for a part of the cloth; (that is, the ratio of \$72 to \$16,) is $\frac{72}{16} = \frac{9}{2}$. And so also the ratio of the cloth bought, to the cloth sold; (that is, the ratio of 27 yards to 6 yards,) is $\frac{27}{6}$ which equals $\frac{9}{2}$. Here we see, that although the ratios are differently expressed, they are, notwithstanding, equal to one another.

§ 111. When the ratio of two quantities is equal to the ratio of other two quantities, there is said to be a PROPORTION between them; that is, *an equality of ratios is called a proportion.*

§ 112. Our chief business with ratios at present, is to learn when they form a proportion; that is, when they are equal to one another. Now, as they may be expressed in the form of a fraction; it is evident, that when they are brought to a common denominator, if the fractions are equal their numerators will be the same, and if they are not equal their numerators will not be the same.

For example, is 11 to 21 = 33 to 63? We pursue our inquiry as follows: 11 to 21 is the same as $\frac{11}{21}$, and 33 to 63 is the same as $\frac{33}{63}$. We bring the fractions to a common denominator by § 105.

$$\frac{11}{21} \text{ and } \frac{33}{63} = \frac{693}{1323} \text{ and } \frac{693}{1323}.$$

We find they are equal, and the four terms 11 to 21 = 33 to 63 are proportional.

§ 113. Although ratios are sometimes expressed fractionally, they are generally expressed as follows: 11 : 21 and 33 : 63; that is, 11 divided by 21, 33 divided by 63. The pupil will see that we employ the same sign that expresses division, with the exception of the — between the two dots

Question. Can a ratio be equal to another, and yet be differently expressed? Give an example. What do we call an equality of ratios? How may we determine whether ratios are equal? Give an example. Are ratios always expressed by fractions? How else?

The sign $:$ is read *is to*, and the foregoing examples are read 11 *is to* 21 and 33 *is to* 63.

§ 114. When four quantities are proportional, they are written thus, $11 : 21 :: 33 : 63$. The sign $::$ is read *as*; and the whole expression is read, 11 is to 21 as 33 is to 63.

§ 115. In a proportion, the first and the last terms are called *extremes*, and the two middle terms are called *means*. In the above proportion, 11 and 63 are the extremes, and 21 and 33 are the means.

§ 116. In order to derive any important use from a proportion, we wish the pupil to recollect the method employed to find whether four quantities are proportional. We multiplied (see § 112) the first numerator by the last denominator, to find *one* new numerator. These were the two *extremes*. We also multiplied the last numerator by the first denominator, to find the *other* new numerator. These were the two *means*. And hence we learn, that *if four quantities are proportional, the product of the two extremes is equal to the product of the means*.

RULE.

§ 117. *A proportion may be reduced to an equation by multiplying the extremes together for one member; and multiplying the means together for the other member.* Thus, $2 : 7 :: 8 : x$, becomes in an equation $2x = 56$; whence $x = 28$.

Or, *the fourth term may be found by multiplying the two means together, and dividing their product by the first extreme.*

Questions. How is a proportion written? Which terms are the extremes? Which terms are the means? In bringing the ratios to a common denominator, what did we do with the extremes? What did we do with the two means? What principle does this show? What then can we do with a proportion?

EQUATIONS.—SECTION 10.

1. If you divide \$75 between two men in the proportion of 3 to 2, what will each man receive?

Stating the question, x = the share of one.

$75 - x$ = the share of the other.

Making the proportion. $x : 75 - x :: 3 : 2$

Multiplying ext. and means }
to reduce to an equation, } $2x = 225 - 3x$

Transposing and uniting, $5x = 225$

Dividing, $x = 45$

Ans. \$45; and \$30.


2. Divide \$150 into two parts, so that the smaller may be to the greater as 7 to 8. Ans. 70; and 80.

3. Divide \$1235 between A and B, so that A's share may be to B's as 3 to 2. Ans. A's share \$741; B's \$494.

4. Two persons buy a ship for \$8640. Now, the sum paid by A is to that paid by B, as 9 to 7. What sum did each contribute? Ans. A paid \$4860; B \$3780.

5. A prize of \$2000 was divided between two persons, whose shares were in proportion as 7 to 9. What was the share of each? Ans. \$875; and \$1125.

6. A gentleman is now 30 years old, and his youngest brother 20. In how many years will their ages be as 5 to 4?

 After stating the question, the proportion will be $30 + x : 20 + x :: 5 : 4$. Ans. 20 years.

7. What number is that, which, when added to 24, and also to 36, will produce sums that will be to each other as 7 to 9? Ans. 18.

8. Two men commenced trade together. The first put in


\$40 more than the second; and the stock of the first was to that of the second as 5 to 4. What was the stock of each?

Ans. \$200; and \$160.

9. A gentleman hired a servant for \$100 a year, together with a suit of clothes which he was to have immediately. At the end of 8 months, the servant went away, and received \$60 and kept the suit of clothes. What was the value of the suit of clothes?

Ans. \$20.


10. A ship and a boat are descending a river at the same time; and when the ship is opposite a certain fort, the boat is 13 miles ahead. The ship is sailing at the rate of 5 miles, while the boat is going 3. At what distance below the fort will they be together?

 The ship sails x miles from the fort; the boat will sail 13 miles less.

Ans. $32\frac{1}{2}$ miles.

§ 118. It is very often the case that a problem is easily solved by using simply the *ratio*, instead of a proportion.

Operation by Ratio.

 In these questions the pupil must not use *any* proportions.

11. Divide 40 apples between two boys in the proportion of 3 to 2.

Stating the question,

x = the share of one.

Now, as the ratio of the first to the second is $\frac{3}{2}$; then the ratio of the second to the first is $\frac{2}{3}$. Therefore,

$$\frac{2x}{3} = \text{the share of the second.}$$

Forming the equation, $x + \frac{2x}{3} = 40$

Multiplying by 3, $3x + 2x = 120$

Uniting terms, $x = 24$

Ans. 24, and 16.

12. Three men trading in company, gain \$780. As often as A put in \$2, B put in \$3, and C put in \$5. What part of the gain must each of them receive?

Stating the question,

$$x = \text{A's share.}$$

$$\frac{3x}{2} = \text{B's share.}$$

$$\frac{5x}{2} = \text{C's share.}$$

Forming the equation,
$$x + \frac{3x}{2} + \frac{5x}{2} = 780.$$

Ans. A, \$156; B, \$234; C, \$390.


13. Two butchers bought a calf for 40 shillings, of which the part paid by A, was to the part paid by B, as 3 to 5. What sum did each pay? Ans. A paid 15s.; B, 25s.

14. Divide 560 into two such parts, that one part may be to the other as 5 to 2. Ans. 400, and 160.

15. A field of 864 acres is to be divided among three farmers, A, B, and C; so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much must each receive? Ans. A, 135; B, 297; C, 432 acres.

16. Three men trading in company, put in money in the following proportion; the first 3 dollars as often as the second 7, and the third 5. They gain \$960. What is each man's share of the gain? Ans. \$192; \$448; \$320.

17. Find two numbers in the proportion of 2 to 1, so that if 4 be added to each, the two sums will be in proportion of 3 to 2.

 The last expression means that the greatest is $\frac{3}{2}$ of the smallest; or the smallest is $\frac{2}{3}$ of the greatest. Ans. 8 and 4.

18. Two numbers are to each other as 2 to 3; but if 50 be subtracted from each, one will be one-half of the other. What are the numbers? Ans. 100 and 150.

19. A sum of money is to be divided between two persons, A and B; so that as often as A takes \$9, B takes \$4. Now

it happens that A receives \$15 more than B. What is the share of each? Ans. A, \$27; B, \$12.

20. There are two numbers in proportion of 3 to 4; but if 24 be added to each of them, the two sums will be in the proportion of 4 to 5. What are the numbers? Ans. 72 and 96.

21. A man's age when he was married was to that of his wife as 3 to 2; and when they had lived together 4 years, his age was to hers as 7 to 5. What were their ages when they were married? Ans. His age, 24; hers, 16 years.

22. A certain man found when he married, that his age was to that of his wife as 7 to 5. If they had been married 8 years sooner, his age would have been to hers as 3 to 2. What were their ages at the time of their marriage?

Ans. His age, 56 years; hers, 40.

23. A man's age, when he was married, was to that of his wife as 6 to 5; and after they had been married 8 years, her age was to his as 7 to 8. What were their ages when they were married? Ans. Man, 24; wife, 20 years.

24. A bankrupt leaves \$8400 to be divided among four creditors, A, B, C, and D, in proportion to their claims. Now, A's claim is to B's as 2 to 3; B's claim to C's as 4 to 5; and C's claim to D's as 6 to 7. How much must each creditor receive?

Ans. A, \$1280; B, \$1920; C, \$2400; D, \$2800.

25. A sum of money was divided between two persons, A and B, so that the share of A was to that of B as 5 to 3. Now, A's share exceeded $\frac{5}{9}$ of the whole sum by \$50. What was the share of each? Ans. \$450, and \$270.

In generalization, see page 154

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SECTION XI.

EQUATIONS WITH TWO UNKNOWN QUANTITIES.

§ 119. It frequently happens, that *several* unknown quantities are introduced into a problem. But when this is the case, if the conditions will give rise to as many equations, independent of each other, as there are unknown quantities, there is no difficulty in finding the value of each quantity.

§ 120. An equation is said to be independent of another when it cannot, either by multiplication or by division, be changed into that other, Thus, $7x - y = 47$, is independent of the equation $10y + 4x = 50$; because one of them cannot be so altered as to make the other. But, $7x - y = 47$, is not independent of the equation $21x - 3y = 141$; because the last is made by multiplying the first by 3.

§ 121. At present we will attend to those equations only that include *two* unknown quantities, *each* represented by a *different letter from the other*.

§ 122. In equations that contain two unknown quantities, our first object must be to find the value of one of them; and in order to do this, the preliminary step is to derive from the equations that are given, another equation which shall have but one unknown quantity. This operation is called *eliminating*, or *exterminating the other unknown quantities*.

§ 123. There are three different methods of forming *one* equation with *one unknown quantity* from two equations containing two unknown quantities. With each of these, the learner should become familiar; as it is sometimes convenient to use one of them, and sometimes another.

Questions. When is one equation said to be independent of another? Explain. When there are several unknown quantities, how many independent equations are necessary to solve the question? What is it to eliminate or exterminate an unknown quantity?

FIRST METHOD OF EXTERMINATION.

§ 124. It is necessary here to recollect what was stated in § 50 and § 53, that *when equals are added to equals, their sums will be equal*; and also, *when equals are subtracted from equals, the remainders are equal*.

Thus, suppose we have the equation $x + 14 = 36$, and suppose also that we know that $y = 8$; then if we will add the two first members together, and also the two last, the members will still be equal to one another, as follows: $x + 14 + y = 36 + 8$.

And also if we subtract y from the first member, and 8 from the second, the members will still be equal to one another; thus, $x + 14 - y = 36 - 8$.

§ 125. This principle can be easily applied for the extermination of unknown quantities. For, if in both of two equations, *one of the unknown quantities has the same co-efficient*, but after *different signs*; it is evident that if we add both equations together, viz. the first member to the first member, and the last member to the last member; and then unite terms, we shall cancel the two quantities that are alike with different signs; a new equation will be formed, in which that unknown quantity will disappear.

EXAMPLES.

1. Given the two $\begin{cases} 3x + 2y = 26 \\ 5x - 2y = 38 \end{cases}$ to find the values of equations, x and y .

Adding together the two right hand members, and also the two left; we have the equation

$$3x + 2y + 5x - 2y = 26 + 38$$

Uniting terms and canceling $2y$, $8x = 64$

Dividing, $x = 8$

Questions. What is the effect of adding two equations together, (the first member to the first member, and the last member to the last member)? In what case can an unknown quantity be exterminated by adding two equations together?

Now, if $x = 8$; then in the first equation, how much will $3x$ equal?

The first equation will then become $24 + 2y = 26$,
from which we may find the value $y = 1$.

2. Given the equations, $\begin{cases} 7x + 4y = 58 \\ 9x - 4y = 38 \end{cases}$ to find the values of x and y .


Ans. $x = 6$.

Question. Why do you add the equations?

If $x = 6$, what does $7x$ in the first equation equal? Then what does y equal?

Ans. $y = 4$.

3. Given the equations, $\begin{cases} 5x + 6y = 58 \\ 2x + 6y = 34 \end{cases}$ to find the values of x and y .

 In this example, it is plain that we cannot destroy the y 's by adding them together. But we have before seen, § 62, that if all the signs are changed, the equation will not be affected. Let us then change the signs of the second equation, so that the y 's may have different signs. The two equations

will then become $\begin{cases} 5x + 6y = 58 \\ -2x - 6y = -34 \end{cases}$ which,

when added together, become $5x + 6y - 2x - 6y = 58 - 34$

Uniting terms,

$$3x = 24$$

Therefore,

$$x = 8.$$

Question. What operation is performed by changing signs?

§ 126. In the last example, if we take the equations before the alteration of the second, thus, $\begin{cases} 5x + 6y = 58 \\ 2x + 6y = 34 \end{cases}$ and subtract the second from the first, the result will be the same as it was by changing the signs and adding. As follows:

$$5x + 6y - 2x - 6y = 58 - 34.$$

Whence we learn that, if in both equations one of the unknown quantities has the *same co-efficient* and also the *same sign*; and we subtract one equation from the other, (viz. the first member from the first member, and the second member

Questions. In what case can an unknown quantity be exterminated by subtracting one equation from another? Explain the reason.

from the second member,) and unite the terms; we shall form a new equation in which that unknown quantity will disappear.


§ 127. It is evident that we may *suppose* the signs changed; and so unite the terms immediately, without actually writing the whole work. This must be done hereafter.

4. Given the equations; $\begin{cases} 6x+7y=79 \\ 6x+3y=51 \end{cases}$ to find the values of x and y .

Subtracting second from first, and unite. $\begin{cases} \\ 4y=28 \end{cases}$

Ans. $x=5$; $y=7$.

5. Given $\begin{cases} 5x+6y=64 \\ 2x+6y=58 \end{cases}$ to find x and y .


 In this example, the y 's are alike in both equations; and are therefore called *identical terms*. As they have the *same sign*, in order to cancel them we subtract the second from the first, and unite.

$$\begin{array}{r} 5x+6y=64 \\ 2x+6y=58 \\ \hline 3x \quad = 6 \end{array}$$

Ans. $x=2$; $y=9$.

6. Given $\begin{cases} x+3y=73 \\ x+6y=106 \end{cases}$ to find x and y .

Which are the identical terms in this example?


 To cancel the x 's, subtract the upper from the lower.

Ans. $x=40$; $y=11$.

7. Given $\begin{cases} 12x+8y=92 \\ 12x-21y=63 \end{cases}$ to find x and y .

Ans. $x=7$; $y=1$.

8. Given $\begin{cases} 2x+2y=18 \\ 3x-2y=7 \end{cases}$ to find x and y .

 As the signs of the identical terms are unlike, to cancel them we add the second to the first, and unite.

$$\begin{array}{r} 2x+2y=18 \\ 3x-2y=7 \\ \hline 5x \quad = 25 \end{array}$$


Ans. $x=5$; $y=4$.

9. Given $\begin{cases} 4x+3y=22 \\ -4x+2y=-12 \end{cases}$ to find x and y .

Ans. $x=4$; $y=2$.


Questions. Is it necessary to write out the whole work? What are *identical terms*?

10. Given $\begin{cases} 3x+5y=40 \\ x+2y=14 \end{cases}$ to find x and y .

 In this example, neither of the unknown quantities has the same co-efficient in both equations. But both members of the last equation can be multiplied by 3, without destroying the equality, § 82; and then the co-efficients of the x 's will be alike in both equations. Thus, $\begin{cases} 3x+5y=40 \\ 3x+6y=42 \end{cases}$

Ans. $x=10$; $y=2$.


11. Given $\begin{cases} 6x+5y=128 \\ 3x+4y=88 \end{cases}$ to find x and y .

 Multiply the second by 2. Ans. $x=8$; $y=16$.

12. Given $\begin{cases} x+2y=17 \\ 3x-y=2 \end{cases}$ to find x and y .

 Make y 's identical. Ans. $x=3$; $y=7$.

13. Given $\begin{cases} 2x-y=6 \\ 4x+3y=22 \end{cases}$ to find x and y .

 Multiply the first by 2 and the x 's will be identical.

Ans. $x=4$; $y=2$

14. Given $\begin{cases} 2x+3z=38 \\ 6x+5z=82 \end{cases}$ to find x and z .

Ans. $x=7$; $z=8$

15. Given $\begin{cases} 4x+6y=46 \\ 5x-2y=10 \end{cases}$ to find x and y .

Ans. $x=4$; $y=5$.

16. Given $\begin{cases} 2x+3y=31 \\ 4x-3y=17 \end{cases}$ to find x and y .

Ans. $x=8$; $y=5$.


17. Given $\begin{cases} 4y+z=102 \\ y+4z=48 \end{cases}$ to find y and z .

Ans. $y=24$; $z=6$.

18 Given $\begin{cases} 2x+3y=7 \\ 8x-10y=6 \end{cases}$ to find x and y .

Ans. $x=2$; $y=1$.

19. Given $\begin{cases} 5y+3x=93 \\ 3y+4x=80 \end{cases}$ to find y and x .

 In this example, we cannot obtain identical terms by one

multiplication. But we may apply the same principle, § 105, that is used for finding a common denominator. For, if we multiply the co-efficient of the first equation by the co-efficient of the second, the product will be the same as if we multiply the co-efficient of the second by the co-efficient of the first.

Thus,

$$\begin{array}{rcl}
 \text{Multiplying the first by 4,} & 20y + 12x = 372 \\
 \text{Multiplying the second by 3,} & 9y + 12x = 240 \\
 \text{Subtracting the second from } \left. \begin{array}{l} \text{the first, and unite,} \end{array} \right\} & \begin{array}{r} 11y = 132 \\ \text{Ans. } y = 12; x = 11. \end{array}
 \end{array}$$

20. Given $\left\{ \begin{array}{l} 4y - 5z = 2 \\ 5y - 4z = 7 \end{array} \right\}$ to find y and z .

$$\begin{array}{rcl}
 \text{Multiplying the first by 5,} & 20y - 25z = 10 \\
 \text{Multiplying the second by 4,} & 20y - 16z = 28 \\
 \text{Subtracting the first from the } \left. \begin{array}{l} \text{second, and unite,} \end{array} \right\} & \begin{array}{r} 9z = 18 \\ \text{Ans. } y = 3; z = 2. \end{array}
 \end{array}$$

§ 128. From the foregoing, we derive the following:

RULE I. TO EXTERMINATE AN UNKNOWN QUANTITY.

First, *Transpose*, so as to bring both of the unknown quantities to the left; x 's under x 's; y 's under y 's, &c.

Determine which of the unknown quantities you will exterminate; and then, if it is necessary, multiply or divide one or both of the equations so as to make the term which contains that unknown quantity to be the same in both.

Then if the identical terms have LIKE signs in both equations, SUBTRACT one equation from the other; but if they have UNLIKE signs, ADD one equation to the other. And the result will be an equation containing only one unknown quantity.

Questions. What is the first operation for exterminating an unknown quantity? Repeat the whole rule. What are identical terms? Why do we *add* them when the *signs* are unlike? Why do we *subtract* them when the signs are alike?

EQUATIONS.—SECTION 11.

1. What two numbers are those whose sum is 20 and difference 12?

Stating the question,

x = greater number.

y = the less.

Then forming the equations,

$$x + y = 20$$

$$x - y = 12$$

As the signs of the identical terms }
are unlike, add the equations, } $2x = 32 \therefore x = 16$

Substituting 16 for x in the first, $16 + y = 20$

Transposing and uniting, $y = 4$.

Ans. 16 and 4.

2. A market woman sells to one person, 3 quinces and 4 melons for 25 cents; and to another, 4 quinces and 2 melons, at the same rate, for 20 cents. How much are the quinces and melons apiece?

After the statement, forming the }
equations, } $3x + 4y = 25$

Multiplying the second by 2, $4x + 2y = 20$

Subtracting first from third, $8x + 4y = 40$

$$5x = 15$$

Ans. Quinces, 3 cents apiece; melons, 4.

In our solutions after this, we shall number the lines, so that any reference to them will be easily understood.

3. A man bought 3 bushels of wheat and 5 bushels of rye for 38 shillings; and at another time, 6 bushels of wheat and 3 bushels of rye for 48 shillings. What was the price for a bushel of each?

Let x = price of wheat, and y = price of rye.

1. By the first condition, $3x + 5y = 38$

2. By the second, $6x + 3y = 48$

3. Multiply the 1st by 2, $6x + 10y = 76$

4. Subtracting the 2d from the 3d, $7y = 28 \therefore y = 4$


5. Substituting 4 for y in the 1st, $3x + 20 = 38$.

Ans. Wheat for 6s.; rye for 4s

4. Two purses together contain \$400. If you take \$40 out of the first and put them into the second, then there is the same in each. How many dollars does each contain?

Let x = the number in the first.

y = the number in the second.

 Although in these examples we have omitted the statement, it is expected the pupil will state them as usual.

1. By the first condition, $x + y = 400$

2. By the second, $x - 40 = y + 40$

3. Transposing the 2d, $x - y = 80$

4. Adding the 1st to the 3d, $2x = 480 \therefore x = 240$

Ans. The first, \$240; the second, \$160.

5. A gentleman being asked the age of his two sons, replied, that if to the sum of their ages 25 be added, this sum will be double the age of the eldest; but if 8 be taken from the difference of their ages, the remainder will be the age of the youngest. What is the age of each?

Let x = the age of the eldest. y = the age of the youngest.

1. By the first condition, $x + y + 25 = 2x$

2. By the second, $x - y - 8 = y$

3. Transposing and uniting 1st, $-x + y = -25$

4. Transposing and uniting 2d, $x - 2y = 8$

5. Adding the 3d and 4th, $-y = -17$

6. Substituting 17 for y in the 3d, $-x + 17 = -25$

7. Transposing, $-x = -42$

Ans. Eldest, 42; youngest, 17.

6. A gentleman paid for 6 pair of boots and 4 pair of shoes \$44; and afterwards for 3 pair of boots and 7 pair of shoes, \$32. What was the price of each pair?

Ans. Boots, \$6; shoes, \$2.

7. A man spends 30 cents for apples and pears, buying his apples at the rate of 4 for a cent, and his pears at the rate of 5 for a cent. He afterwards let his friend have half of his apples and one-third of his pears for 13 cents, at the same rate. How many did he buy of each sort?

Let x = number of apples.

y = number of pears.

$\frac{1}{4}$ cent = price of 1 apple.

$\frac{1}{5}$ cent = price of 1 pear.

$\frac{x}{4}$ cents = price of all the apples

$\frac{y}{5}$ cents = price of all the pears.

1. By the first condition,

$$\frac{x}{4} + \frac{y}{5} = 30$$

2. By the second,

$$\frac{x}{8} + \frac{y}{15} = 13$$

3. Dividing the 1st by 3,

$$\frac{x}{12} + \frac{y}{15} = 10$$

4. Subtracting 3d from 2d,

$$\frac{x}{8} - \frac{x}{12} = 3$$

5. Multiplying by 24,

$$3x - 2x = 72 \therefore x = 72.$$

Ans. 72 apples ; 60 pears.

8. One day a gentleman employs 4 men and 8 boys to labor for him, and pays them 40s. ; the next day he hires at the same rate, 7 men and 6 boys, for 50s. What are the daily wages of each ?


Ans. Man's, 5s. ; boy's, 2s. 6d.

9. It is required to find two numbers with the following properties : $\frac{1}{2}$ of the first with $\frac{1}{3}$ of the second shall make 16, and $\frac{1}{4}$ of the first with $\frac{1}{5}$ of the second shall make 9.

 Performed as Problem 7.

Ans. 12 and 30.

10. Says A to B, Give me 5s. of your money, and I shall have twice as much as you will have left. Says B to A, Give me 5s. of your money, and I shall have three times as much as you will have left. What had each ?

 In the equations, first transpose so that x shall be under x , and y under y

Ans. A, 11s. ; B, 13s.

11. Two men agree to buy a house for \$1200. Says A to B, Give me $\frac{2}{3}$ of your money, and I shall be able to pay for it all; No, says B, give me $\frac{3}{4}$ of yours, and then I can pay for it. How much money had each? Ans. A, \$800; B, \$600.

12. Find two numbers with the following properties: The products of the first by 2, and the second by 5, when added, are equal to 31; also, the products of the first by 7, and the second by 4, when added, are equal to 68. Ans. 8 and 3.


13. A paid B 20 guineas, and then B had twice as much money as A had left; but if B had paid A 20 guineas, A would have had three times as much as B had left. What sum did each possess at first? Ans. A, 52 guineas; B, 44.

14. A person has a saddle worth £50, and two horses. When he saddles the poorest horse, the horse and saddle are worth twice as much as the best horse; but when he saddles the best, he with the saddle is worth three times the poorest. What is the value of each horse?

Ans. Best, £40; poorest, £30.

15. A merchant sold a yard of broadcloth and 3 yards of velvet for \$25; and, at another time, 4 yards of broadcloth and 5 yards of velvet for \$65. What was the price of each per yard?

Ans. Broadcloth, \$10; velvet, \$5.

16. A person has 500 coins, consisting of eagles and dimes; and their value amounts to \$1931. How many has he of each coin?  The solution must be in *cents*.

Ans. 190 eagles; 310 dimes.

17. In the year 1299, three fat oxen and six sheep together cost 79 shillings; and the price of an ox exceeded the price of 12 sheep by 10 shillings. What was the value of each?

Ans. An ox, 24s.; a sheep, 1s. 2d.

18. Two persons talking of their ages, A says to B, 8 years ago I was three times as old as you were; and 4 years hence, I shall be only twice as old as you will be. What are their present ages?

Ans. A, 44; B, 20 years.

19. A farmer sold to one man 30 bushels of wheat and 40 of barley for 270 shillings; and to another, 50 bushels of wheat and 30 of barley for 340 shillings. What was the price per bushel of each? Ans. Wheat, 5s.; barley, 3s.

20. A man and his wife and child dine together at an inn. The landlord charged 15 cents for the child, and for the woman he charged as much as for the child and $\frac{1}{3}$ as much as for the man; but for the man he charged as much as for the woman and child together. What did he charge for each?

Ans. 45 cents for the man; and 30 cents for the woman.

21. A gentleman has two horses, and also a chaise worth \$250. If the first horse be harnessed, he and the chaise will be worth twice as much as the second horse; but if the second be harnessed, he and the chaise will be worth three times as much as the first horse. What is the value of each horse?

Ans. First, \$150; second, \$200.

22. A is in debt \$1200, and B owes \$2500; but neither has enough to pay his debts. A says to B, Lend me the $\frac{1}{8}$ of your fortune, and then I can pay my debts. But B answered, Lend me the $\frac{1}{9}$ of your fortune, and I can pay my debts. What was the fortune of each?

Ans. A, \$900; B, \$2400.

23. A wine merchant has two kinds of wine, one at 5s. a gallon, and the other at 12s.; of which he wishes to make a mixture of 20 gallons that shall be worth 8s. a gallon. How many gallons of each sort must he use?

Ans. $8\frac{4}{7}$ gallons of that at 12s.; $11\frac{3}{7}$ of that at 5s.

SECTION XII.

SECOND METHOD OF EXTERMINATION.

§ 129. In each of the preceding questions, we first found the value of *one* of the unknown quantities; and then substituted that value for that unknown quantity in one of the equations, in order to find the value of the other unknown quantity. This mode of operating furnishes a hint that leads us to another method of extermination.

Let us take the first question in the last section, [p. 88,] in which we have the equations, $\begin{cases} x+y=20 \\ x-y=12 \end{cases}$

The last part of our operation was to substitute the value of x for x itself, in one of the equations. It is evident that we could make this substitution just as well if the value of x was a *literal* quantity, instead of 16. Thus, supposing x to be equal to $\frac{y}{2}$; then substituting it for x , the first equation would be $\frac{y}{2} + y = 20$.

§ 130. Let us therefore transpose the first equation to find what x will equal, just as if we knew the value of y . We shall find that $x = 20 - y$. And then in the second equation, we shall use the value of x instead of x itself.

$$\text{Thus, } 20 - y - y = 12.$$

Transposing and uniting, $-2y = -8 \therefore y = 4$; which was our answer by the first method. Then x will be found by substituting 4 for y . Whence we derive

RULE II. TO EXTERMINATE AN UNKNOWN QUANTITY.

§ 131. *Select the most simple term of the unknown quantities, and by the equation that contains it, find the value of that unknown quantity, as if the other were known; and*

then, in the other equation, substitute this value for the unknown quantity itself.

We shall then have an equation with only *one* unknown quantity ; which may be solved as usual.

EQUATIONS.—SECTION 12.

1. There are two numbers whose sum is 100 ; and three times the less taken from twice the greater, leaves 150 remainder. What are those numbers ?

Let x = greater.

y = less.

$2x - 3y$ = the required subtraction.

Forming the equations, $\left\{ \begin{array}{l} 1. \text{ By the first condition, } x + y = 100 \\ 2. \text{ By the second, } 2x - 3y = 150 \end{array} \right.$

3. Transposing the 1st, $x = 100 - y$

4. Multiplying the 3d by 2, $2x = 200 - 2y$

5. Substituting $200 - 2y$ for $2x$ in the 2d, $\left. \begin{array}{l} \\ \end{array} \right\} 200 - 2y - 3y = 150$

6. Transposing and uniting, $-5y = -50 \therefore y = 10$

7. Substituting 10 for y in the 1st, $x + 10 = 100$

8. Transposing and uniting, $x = 90$

Ans. Greater, 90 ; less, 10.

2. The ages of a father and his son amounted to 140 years ; and the age of the father was to the age of the son as 3 to 2. What were their ages ?

Let x = age of the father.

y = age of the son.

1. By the first condition, $x + y = 140$

2. By the second, $y = \frac{2x}{3}$

3. Substituting $\frac{2x}{3}$ for y in the 1st, $x + \frac{2x}{3} = 140$

4. Multiplying by 3, $3x + 2x = 420$

5. Uniting and dividing, $x = 84$

6. Substituting 84 for x in 1st, $84 + y = 140$

Ans. Father, 84 years ; son, 56

3. Find two numbers, such that $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second shall be 87; and $\frac{1}{5}$ of the first and $\frac{1}{6}$ of the second shall be 55.

Ans. 135, and 168.

4. A says to B, Give me 100 of your dollars, and I shall have as much as you. B replies, Give me 100 of your dollars, and I shall have twice as much as you. How many dollars has each?

Ans. A, \$500; B, \$700.

5. There are two numbers, such that $\frac{3}{4}$ of the first and $\frac{2}{7}$ of the second added together, will make 12; and if the first be divided by 2, and the second multiplied by 3, $\frac{2}{3}$ of the sum of these results will be 26.

Ans. 15, and $10\frac{1}{2}$.

6. Find two numbers in the proportion of 2 to 1, so that if 4 be added to each, their two sums shall be in proportion of 3 to 2.

Ans. 8, and 4.

7. A and B owned 9800 acres of western land. A sells $\frac{1}{4}$ of his, and B sells $\frac{1}{3}$ of his; and they then have just as much as each other. How many acres had each?

Ans. A, 4800; B, 5000.

8. A son asking his father how old he was, received the following reply: My age, says the father, 7 years ago, was four times as great as yours at that time; but 7 years hence, if you and I live, my age will be only double of yours. What was the age of each?

Ans. Father's, 35 years; son's, 14 years.

8. The weight of the head of Goliath's spear was less by one pound than $\frac{1}{8}$ the weight of his coat of mail; and both together weighed 17 pounds less than ten times the spear's head. What was the weight of each?

Ans. Coat, 208 pounds; spear's head, 25 pounds.

10. A market woman bought eggs, some at the rate of 2 for a cent, and some at the rate of 3 for 2 cents, to the amount of 65 cents. She afterwards sold them all for 120 cents, thereby gaining half a cent on each egg. How many of each kind did she buy?

Ans. 50 of the first kind; 60 of the other kind

11. Says A to B, $\frac{1}{3}$ of the difference of our money is equal to yours; and if you give me \$2, I shall have five times as much as you. How much has each? Ans. A, \$48; B, \$12.

12. A and B possess together property to the amount of \$5700. If A's property were worth three times as much as it is, and B's five times as much as it is, then they both would be worth \$23,500. What is the worth of each?

Ans. A, \$2500; B, \$3200.

13. A gentleman has two silver cups, and a cover adapted to each which is worth \$20. If the cover be put upon the first cup, its value will be twice that of the second; but if it be put upon the second, its value will be three times that of the first. What is the value of each cup?

Ans. First cup, \$12; second, \$16.

14. Two men driving their sheep to market, A says to B, Give me one of your sheep, and I shall have as many as you. B says to A, Give me one of your sheep, and I shall have twice as many as you. How many had each?

Ans. A, 5 sheep; B, 7.

15. What two numbers are those, whose difference is 4, and 5 times the greater is to 6 times the less, as 5 to 4?

Ans. 8 and 12.

16. There are two numbers such that $\frac{1}{2}$ of the greater added to $\frac{1}{3}$ of the less, will equal 13; and if $\frac{1}{2}$ of the less be taken from $\frac{1}{3}$ of the greater, the remainder is nothing. What are the numbers?

Ans. 18 and 12.

SECTION XIII.

THIRD METHOD OF EXTERMINATION.

§ 132. The method of substitution as explained in the last section, may be modified a little. We will show how, by using question 1st, in the last section of equations.

The two equations were $\begin{cases} x + y = 100 \\ 2x - 3y = 150 \end{cases}$

We transpose the 1st; thus, $x = 100 - y$.

Now, before we substitute the value of x for x itself in the second equation, we will transpose the second equation so as to make x stand alone; thus, $2x = 150 + 3y$.

Then substitute the value of x as found before by the first equation, $200 - 2y = 150 + 3y$ with which we may proceed as before.

§ 133. Before we make the substitution after transposing, it is generally best to find the value of x *alone* in the second equation. Thus,

Given $\begin{cases} 2x + 3y = 23 \\ 5x - 2y = 10 \end{cases}$ to find x and y .

Transposing and dividing the 1st, $x = \frac{23 - 3y}{2}$

Transposing and dividing the 2d, $x = \frac{10 + 2y}{5}$.

Now, as it is evident that *things which are equal to the same, are equal to one another*; one value of x is equal to the other value of x ; thus,

$$\frac{23 - 3y}{2} = \frac{10 + 2y}{5}$$

Destroying the fractions, $115 - 15y = 20 + 4y$

Transposing, uniting, and dividing, $y = 5$

By substituting the value of y in one of the equations, we find $x = 4$. Whence we derive

RULE III. TO EXTERMINATE AN UNKNOWN QUANTITY.

§ 134. Find by each of the equations, the value of that unknown quantity which is the least involved; and then form a new equation by making one of these values equal to the other.

EQUATIONS.—SECTION 13.

1. Divide \$60 between A and B, so that the difference between A's share and 31, may be to the difference between 31 and B's share, as 6 to 7.

Let x = A's share; and y = B's.

1. By the first condition, $x + y = 60$
 2. By the second, $x - 31 : 31 - y :: 6 : 7$
 3. Multiplying extremes and means, $7x - 217 = 186 - 6y$
 4. Transposing the 1st, $x = 60 - y$
 5. Transposing and uniting the 3d, $7x = 403 - 6y$
 6. Multiplying the 4th, $7x = 420 - 7y$
 7. Equating 5th and 6th, $403 - 6y = 420 - 7y$
 8. Transposing and uniting, $y = 17$
 9. Substituting 17 in the 4th, $x = 60 - 17 = 43$
- Ans. A's share, \$43; B's, \$17.

2. There is a fraction, such that if 1 is added to the numerator, its value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$. What is that fraction?

Let x = numerator; and y = denominator.

The fraction will be, $\frac{x}{y}$

1. By the first condition, $\frac{x+1}{y} = \frac{1}{3}$
 2. By the second, $\frac{x}{y+1} = \frac{1}{4}$
 3. Multiplying the 1st by y , and by 3, $3x+3=y$
 4. Multiplying the 2d by $y+1$, and by 4, $4x=y+1$
 5. Transposing the 4th, $4x-1=y$
 6. Equating 3d and 5th, $3x+3=4x-1$
 7. Transposing and uniting, $-x=-4$
 8. Substituting the value of $3x$ in the 4th, $15=y$
- Ans. $\frac{4}{17}$.

3. There is a certain number, consisting of two places of figures, which is equal to 4 times the sum of its digits; and if 18 be added to it, the digits will be inverted. What is that number?

Let x = first digit or *tens*; and y = the units

$10x + y$ = the number.

$4x + 4y$ = four times the sum of digits.

$10x + y + 18$, = when 18 is added.

$10y + x$, = when the digits are inverted

1. By the first condition, $10x + y = 4x + 4y$

2. By the second, $10x + y + 18 = 10y + x$

3. Transposing and uniting the 1st, $6x = 3y$

4. Transposing and uniting the 2d, $9x = 9y - 18$

5. Multiplying the 4th by $\frac{2}{3}$, $6x = 6y - 12$

6. Equating 3d and 4th, $3y = 6y - 12$

Ans. 24.


4. There is a certain number consisting of two figures; and if 2 be added to the sum of its digits, the amount will be three times the first digit; and if 18 be added to the number, the digits will be inverted. What is the number? Ans. 46.

5. A person has two snuff-boxes and \$8. If he puts the 8 dollars into the first, then it is half as valuable as the other. But if he puts the 8 dollars into the second, then the second is worth three times as much as the first. What is the value of each? Ans. First, \$24; second, \$64.

6. A gentleman has two horses and a chaise. The first horse is worth \$180. If the first horse be harnessed to the chaise, they will together be worth twice as much as the second horse; but if the second horse be harnessed, the horse and chaise will be worth twice and one-half the value of the first. What is the value of the second horse, and of the chaise? Ans. Horse, \$210; chaise, \$240.

7. There is a certain number consisting of two digits. The sum of these digits is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number? Ans. 23.

8. There is a number consisting of two figures. If the number be divided by the sum of the figures, the quotient will be 4; but if the number made by inverting the figures be divided by 1 more than their sum, the quotient will be 6. What is the number?

 In the operation, 4 multiplied by $x+y$, is the same as 4 times $x+y$. Ans. 24.

9. There are two numbers such that the less is to the greater as 2 to 5; and the product made by multiplying the two numbers together, is equal to ten times their sum. What are the numbers?

Let x = the less; and y = the greater.

1. By the first condition,

$$x = \frac{2y}{5}$$

NOTE.—If we wish to multiply y by 4, we put 4 immediately before the y as a co-efficient; and in the same way, if we multiply y by x , we make x the co-efficient of y .

2. By the second,

$$xy = 10x + 10y$$

3. Multiplying the 1st by 10,

$$10x = 4y$$

4. Transposing the 2d,

$$10x = xy - 10y$$

5. Equating 4th and 3d,

$$xy - 10y = 4y$$

NOTE.—When we divide $4x$ by 4, we do it by taking away 4 when we divide $10x$ by 10, we do it by taking away the 10. In the same manner we divide yx by y , in taking away the y .

6. Dividing by y ,

$$x - 10 = 4 \therefore x = 14.$$

7. Substituting 14 for x in the 3d,

$$140 = 4y \therefore y = 35.$$

Ans. 14 and 35.

10. There are two numbers, whose sum is the $\frac{1}{6}$ part of their product; and the greater is to the less as 3 to 2. What are those numbers?

Ans. 15 and 10

SECTION XIV.

EQUATIONS WITH SEVERAL UNKNOWN QUANTITIES.

§ 135. When there are three or more unknown quantities; first, transpose all the unknown quantities to the left, and write them so that letters of the same kind shall be under each other. Then, combine successively one of the equations with each of the others, so as to exterminate the *same* unknown quantity from each. By this means there will be obtained a number of equations one less than the original number. With these perform the same process as before; and proceed in this manner till there is but one equation containing only one unknown quantity; which may be solved by the usual rule. Then by substitution, the value of the other unknown quantities may be found, in the reverse order in which they were exterminated.

EXAMPLES.

1. Given the equations $\left\{ \begin{array}{l} x + y + z = 9 \\ x + 2y + 3z = 16 \\ x - y - 2z = -3 \end{array} \right\}$ to find $x, y,$
and z .

4. Subtracting 1st from 2d, $y + 2z = 7$
5. Subtracting 3d from 1st, $2y + 3z = 12$
6. Multiplying 4th by 2, $2y + 4z = 14$
7. Subtracting 5th from 6th, $z = 2$
8. Substituting value of z in 4th, $y = 3$
9. Substituting in the 1st, $x = 4$

Questions. In solving equations with several unknown quantities, what must be done first? Then which unknown quantity must be exterminated?

2. Given

$$\left\{ \begin{array}{l} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10 \end{array} \right\} \text{ to find } x, \\ y, \text{ and } z.$$

4. Subtracting 1st from 2d,

$$\underline{y + 2z = 33}$$

5. Destroying fractions in 3d,

$$\underline{6x + 4y + 3z = 120}$$

6. Multiplying 1st by 6,

$$\underline{6x + 6y + 6z = 174}$$

7. Subtracting 5th from 6th,

$$\underline{2y + 3z = 54}$$

8. Multiplying 4th by 2,

$$\underline{2y + 4z = 66}$$

9. Subtracting 7th from 8th,

$$\underline{z = 12}$$

Whence by substitution, $y = 9$; and $x = 8$.3. Given $x + y + z = 7$; $2x - y - 3z = 3$; and $5x - 3y + 5z = 19$; to find x, y, z . Ans. $x = 4, y = 2, z = 1$.4. Given $x - y - z = 5$; $3x + 4y + 5z = 52$; and $5x - 4y - 3z = 32$; to find x, y , and z . Ans. $x = 10, y = 3, z = 2$.5. Given $7x + 5y + 2z = 79$; $8x + 7y + 9z = 122$; and $x + 4y + 5z = 55$; to find the values of x, y , and z .

$$\text{Ans. } x = 4, y = 9, z = 3.$$

6. Given $x + y + z = 13$; $x + y + u = 17$; $x + z + u = 18$; and $y + z + u = 21$; to find the values of x, y, z , and u .

$$\text{Ans. } x = 2, y = 5, z = 6, u = 10.$$

PART II.

LITERAL ALGEBRA.

GENERAL PRINCIPLES.

§ 136. THE algebraical operations which we have hitherto treated of, belong to that part of the science which was known to the ancients, and which was in use till about A. D. 1600. About that time, Franciscus Vieta, a Frenchman, introduced the general use of letters into Algebra, (denoting the known quantities in a problem by consonants, and the unknown ones by vowels.)

§ 137. This improvement gave a new aspect to the science. So that now algebra is rather the *representation* of arithmetical results, than the results themselves. And therefore, its most general object is to afford means for investigating the laws of calculation for every description of numerical questions.

§ 138. Operations with numbers cannot furnish general rules, for two reasons. In the first place, we cannot, by mere inspection of the results, determine *how* they were obtained. Thus, 12 may be the result, either of multiplying 3 by 4, or adding 5 to 7, or subtracting 8 from 20, or dividing 48 by 4, &c. &c.

And in the second place, every figure in an arithmetical result has a *determinate* value which is peculiar to itself; and therefore cannot be applied to any other question.

Questions. How does modern algebra differ from the ancient methods? Who introduced the modern method? What is the modern use of algebra? What are the two reasons why we cannot obtain general rules by arithmetical operations?

§ 139. But in algebra, the letters which represent the quantities, retain their identity throughout the whole reasoning; and as the operations of addition, subtraction, multiplication, &c., are only *represented* by signs, we readily see the dependence which the several quantities have upon one another. And as the result is represented by letters, each of which may stand for whatever number we choose, its value is entirely indeterminate; and shows merely what operations it is necessary to perform upon the numbers when a particular value may be assigned to them.

§ 140. Hence, in Literal Algebra, the result does not depend upon the *particular values* of the quantities which we operate with, but rather upon the *nature of the question*; and it will always be the *same* for every question of the *same kind*. The result is therefore a general rule.

§ 141. The principal signs that are used in algebra are the following.

The sign $+$ (*plus*) represents addition. Thus, $a+b$ denotes that b is added to a . $a+b$ is the *sum* of two numbers.

The sign $-$ (*minus*) denotes that the quantity following it is subtracted. Thus, $a-b$ is the remainder obtained by subtracting b from a .

The sign \times (*multiplied by*) denotes that the quantity before it, is multiplied by the quantity that follows it. Thus, $a \times b$ is the product of a and b .

The sign $.$ is sometimes put between two literal quantities instead of \times ; as $a.b$. But more generally, in algebra, the letters are joined together, to represent their product. Thus, ab is the product of two numbers.

The sign \div or $:$ (*divided by*) represent the division of the

Questions. How does algebra differ in these particulars? In literal algebra, what is the use of the answers? Upon what does an algebraical result depend? In what cases are the results the same? Define the signs $+$, $-$, \times , $.$, \div . What is the sum of two numbers, say a and b ? What is their difference? What is their product? What is their quotient?

quantity before it, by the quantity which follows it. Thus, $a \div b$ is a divided by b .

Division is more generally denoted in algebra by writing the divisor under the dividend, in the form of a fraction.

Thus, $\frac{a}{b}$ is the quotient of a divided by b .

The sign $=$ (*equals*) denotes that the whole quantity on the left of it, is equal to the whole quantity on the right of it. Thus, $4 + 8 = 16 - 4$.

The sign \pm or \mp (*plus or minus* and *minus or plus*) shows that either by addition or by subtraction, the effect will be the same. Thus, there are circumstances when $x = \pm a$; that is, x is equal to *plus* a or *minus* a .

The sign \sim or \oslash (*the difference of*) shows that it is not known whether the quantity before it is subtracted from the quantity after it, or the latter is subtracted from the former. Thus, $a \oslash b$ is the difference between a and b , without specifying which is the greatest.

The sign $>$ or $<$ (*greater than* or *less than*) denotes that the quantity towards which it opens is greater than the other. Thus, in $a > b$, a is greater than b .

The sign ——— (*a vinculum*) denotes that all which is put under it, is to be used as one term. Thus, $\overline{a+b-c} \times 3$, signifies that the whole quantity under the vinculum, is to be multiplied by 3.

The line which separates the terms of a fraction is also a vinculum. Thus, $\frac{a+b-c}{3+a}$ signifies that the whole quantity above the line is to be divided by the whole quantity under it.

The $()$ *parenthesis* is frequently used instead of the vinculum; thus, $(a+b) \cdot (c+d)$ signifies the product of $(a+b)$ multiplied by $(c+d)$.

Questions. Define the sign $=$, \pm , \mp , \oslash , $>$, $<$, ——— , $()$. Is the line *above* a quantity the only vinculum? What is the sign of multiplication between vinculum?

The sign ∞ (*infinity*) denotes a quantity that is infinitely large, or a quantity so great that it may be considered larger than any supposable quantity.

The *cipher* 0 is sometimes used to represent a quantity that is less than any quantity that may be mentioned. This is always the case when the cipher is used as a *denominator* of a fraction.

The radical sign $\sqrt{}$ denotes the *root* of the following quantity. When unaccompanied by a figure, it represents the *square* root. But when a figure is put over it, that figure expresses the root that is designed. Thus, $\sqrt[3]{a}$ denotes the 3d root of a . The number over the radical sign is called the *index* of the root.

A *co-efficient* is a number put immediately before a letter, as, $2b$; and denotes how many times the quantity is to be taken.

In such a case, the co-efficient is called a *numeral* co-efficient. Sometimes when one *letter* has been multiplied into another, the first written one is called a *literal* co-efficient; as, in ab , a is the literal co-efficient.

An *exponent* or *index* is a small figure placed a little over and a little to the right of a quantity; as, a^2 . It denotes that the quantity is multiplied by itself; and shows how many times the quantity is to be taken as a factor.

An exponent may be either positive, negative, or fractional; as, a^2 , a^{-2} , $a^{\frac{1}{2}}$. This will be explained in another place.

The sign $::$ represents proportion, and $:$ denotes the ratio of two numbers.

The sign \propto denotes a general proportion.

A *term* is any quantity that is not separated into parts by either of the signs.

Questions. Define the signs ∞ , 0, $\sqrt{}$, $\sqrt[3]{}$. What is the figure over the radical sign called? What is a co-efficient? Is it numeral or literal? What is an exponent or index? What is the sign for proportion? and for ratio? What is the sign for a general proportion? What is a term?

The sign \therefore is used for the word *therefore*

The *hyphen* -- is used for the word *which*; as, $2x = \frac{2}{3}^4$
 $- = 8$; read, *which equals 8*.

A *simple quantity* is that which is represented by one term; as, a , $2b$, -3 , $3mnr$ s, &c.

A *compound quantity* is that which is represented by two or more terms; as, $a + 2ab - x$.

Similar quantities are those which consist of the same letters, or combinations of letters; as, a and $2a$; $6bx$ and $4bx$.

Dissimilar quantities are those which consist of different letters, or different combinations of them; as, a and a^2 ; $2ax$ and $2ab$.

Identical terms are those which are not only similar, but also have the same co-efficient; as, $2ax$ and $2ax$.

Identical expressions are sometimes made up of the same letters, but differently combined in their terms; as, $(3a - 2b)$ and $(2a + a + 5b - 7b)$.

Positive quantities are those which have the sign $+$ before them, either expressed or understood; as, ab , $+ax$.

Negative quantities are those which have the sign $-$ before them; as, -3 , $-2x$.

Given quantities are such as have known values; and are generally represented by the first letters of the alphabet. They are sometimes represented by the initial of the *names* that stand for them; as s for the *sum*; d for the *difference*.

Unknown quantities are those which are to be discovered; and are generally represented by some of the final letters of the alphabet.

A quantity, when represented by one term, is sometimes called a *nomial*. When it has two terms, it is called a *bi-*

Questions. What is the sign for therefore? For which? What is a simple quantity? What is a compound quantity? What are similar quantities? Dissimilar quantities? Identical terms? Positive quantities? Negative quantities? Given quantities? Unknown quantities? What is a nomial? Binomial?

nomial; when it has three, it is called a *trinomial*; when it has many, it is called a *multinomial* or *polynomial*.

Each of the literal factors which compose a term, is called a *dimension* of that term; and the number of these dimensions or factors is called the *degree* of the term. The co-efficient is not counted as a dimension. Thus, $2a$ is a term of one dimension or of the first degree; $6ax$ is a term of two dimensions or of the second degree; $5a^2x^2$ is a term of four dimensions or of the fourth degree. And, generally, the degree of the term is the *sum of the exponents* which belong to its letters.

A *polynomial* is called *homogeneous* when all its terms are of the same degree. Thus, $3abc - ax^2 + c^3$, is homogeneous; but $8a^3 - 4ab + c$ is not homogeneous.

§ 142. The use of the foregoing signs makes algebra a species of language which brings our reasonings into a very small space; so that in solving a problem, or demonstrating the existence of a numerical relation, the connection of the several ideas is perceived with great facility.

Example 1. The sum of \$660 was subscribed for a certain purpose, by two persons, A and B; of which B gave twice as much as A. What did each of them subscribe?

Now, a question similar to this, and with the same numbers, was solved in the First section of Equations, on page 25. We will solve this in the same manner, with the exception of using a instead of 660.

Stating the question,

x = what A gave.

$2x$ = what B gave.

Both together gave $x + 2x$; also they gave a dollars.

Forming the equation,

$$x + 2x = a$$

Uniting the terms,

$$3x = a$$

Dividing by 3,

$$x = \frac{a}{3}$$

Questions. What is a trinomial? Polynomial? What is the dimension of a term? What is the degree of a term? When is a quantity homogeneous? What is the use of algebraic signs?

Here we find that A subscribed $\frac{1}{3}$ of a , which at this time stands for \$660.

But it is very plain that we would solve the question in the very same manner, if the sum were \$240. And in that case a would stand for \$240; and A's share of it would be \$80, and B's share, \$160.

In the same manner, if the sum were \$360; then a would stand for \$360, and A's share would be $\frac{1}{3}$ of \$360; that is, \$120. And in the same manner we may make a represent any sum; and still A's share of it would be $\frac{1}{3}$ of it. Hence, this substitution of a letter for a number, is called *generalizing the operation*.

We see that this result has given us a *general rule* for dividing any sum between two, so that one of them shall have twice as much as the other. The rule is, *The least share shall be one-third of the sum; and the greatest share, two-thirds of it.*

Problems.—In the same manner generalize all the problems in the First section of Equations; page 25.

Example 2. What number is that, which, with 5 added to it, will be equal to 40?

This is the first problem in section 2, which we will generalize; using a for 40, and b for 5.

Stating the question,

x = the number.

$x + b$ = after adding.

Forming the equation,

$x + b = a$

Transposing b ,

$x = a - b$.

We see that the answer is found by subtracting the 5 from the 40. Thus, $40 - 5 = 35$.

Example 3. Generalize problem 3 of section 2. It will be found that the literal answer is the same as in example 2; because the questions are similar. In this example, a represents 23, and b represents 9. Whence $x = 23 - 9 = 14$.

Question. What do we call the generalization of an operation?

Example 4. Divide 17 dollars between two persons, so that one may have \$4 more than the other. [Prob. 4, sec. 2.]

Let 17 be represented by a , and 4 by b .

Stating the question, $x =$ the least share.

$x + b =$ the greater.

$x + x + b =$ both shares.

Forming the equation,

$$x + x + b = a$$

Transposing b ,

$$x + x = a - b$$

Uniting terms,

$$2x = a - b$$

Dividing by 2

$$x = \frac{a - b}{2}, \text{ or, } \frac{a}{2} - \frac{b}{2}$$

The answer is found by subtracting the difference or 4, from the whole sum, and then dividing by 2.

$$\text{Thus, } \frac{17 - 4}{2} = \frac{13}{2} = 6\frac{1}{2}.$$

And this is the rule for all similar sums.

The 5th question, on page 32, is similar to the one just performed; and in that, a represents 55, and b represents the difference or 7. The numerical answer is found by the rule just shown.

$$\text{Thus, } \frac{a - b}{2} = \frac{55 - 7}{2} = \frac{48}{2} = 24.$$

§ 143. As this rule is of some importance, it will be well to remember it. *If, from a number to be divided into two parts, we subtract the difference of those parts, half the remainder will be equal to the smaller part.*

Application. Perform the 7th and 10th by this rule.

Example 5. In the same questions, let us take x for the greatest share. Then $x - b =$ the less.

Forming the equation,

$$x + x - b = a$$

Transposing and uniting,

$$2x = a + b$$

Dividing by 2,

$$x = \frac{a + b}{2}, \text{ or, } \frac{a}{2} + \frac{b}{2}.$$

§ 144. Here we have another rule. *If, to a number to be divided into two parts, we add the difference between those parts, half the sum will be equal to the greater part*

Application. Find the greater part in questions 4, 5, 7, and 10, on page 32, by this rule; *without algebra*.

§ 145. The *mere letters* in the answer of an algebraical operation, form what is called a *formula*; because they are the *form* of the solutions of all similar questions. And this is the advantage of representing the quantities by letters. For, as arithmetical operations on them can only be indicated, the result also must be merely an indication; and this indication will apply to any question, in the enunciation of which the only things which vary, are the *numerical values* of the quantities. Thus, the formula $x = \frac{s+d}{2}$, denotes that the greater share is found by adding the difference to the sum, and dividing the amount by 2.

Example 6. The learner must now generalize problem 6, on page 32; using b and c for the two differences. The formula that he obtains will be the answer for questions 8, 9, 11, and 13. And in each of the five problems the pupil must verify the answer by substituting the given quantities for the letters. Thus, $x = \frac{a-2b-c}{3} = \frac{73-8-5}{3} = 20$.

We have said that in algebra the arithmetical operations on numbers are only *represented* by different methods of combining the signs that stand for those quantities. And now, although we have shown in our progress thus far, what some of those methods are, it may be well to review them a little.

Questions. What two important rules have we found by generalizing? What do we call a formula in algebra? Why? What advantage do we derive from formulas? What then is the real effect of an algebraical result?

I.

ADDITION AND SUBTRACTION OF ALGEBRAICAL QUANTITIES.

§ 146. *One algebraical quantity is added to another by writing one quantity after the other, taking care to preserve to each term its respective sign. Thus, $a+f-c$ is added to $d-e+b$, so as to make $d-e+b+a+f-c$. Or, as it is easier to read the letters in their alphabetical order, their sum may be written $a+b-c+d-e+f$. Again, when $-a$ is added to b , we preserve the sign; thus, $b-a$.*

§ 147. *One algebraical quantity is subtracted from another, by changing the sign or signs of the quantity which is to be subtracted, and then writing that quantity after the other. Thus, $a+h-y$ is subtracted from $b-x+c$, by first making it $-a-h+y$, and then writing the whole quantity, $b-x+c-a-h+y$; or, $b-a+c-h+y-x$.*

§ 148. *After the addition or subtraction has been performed, if there are any similar quantities in the result, they may be united by adding the co-efficients of all the positive similar terms, and affixing their literal part; then adding all the negative similar terms in the same manner; and then subtracting the less sum from the greater, and retaining, in the result, the sign of the greater.*

§ 149. *In uniting the terms of compound numbers, we consider the literal part of the term as a unit; thus, $2a$ and $3a$, are regarded as 2 units and 3 units of a particular kind which when put together, make five units of that kind. Now we have seen, § 141, that the co-efficient of a quantity may*

Questions. How is addition performed in algebra? How is a negative quantity added? How is subtraction performed? How are algebraical quantities united? In determining whether the quantities are similar, what part of the quantity do we examine?

also be literal; as in ba , ca , &c. In such cases, the *whole term* ba or ca becomes a unit, each of a different kind; and of course are not similar quantities, and cannot be united.

§ 150. But if there are several similar units of this kind, they may be united by the general rule. Thus, $ba - ca + ba$ $ca + ba + ca$, can be united into, $3ba + ca$. $ax - bx + ax + 2bx - 3ax + bx$, are equal to, $-ax + 2bx$; or $2bx - ax$.

§ 151. Again, we have seen, § 68, that several quantities are sometimes united by a vinculum. In such cases, all that is embraced by the vinculum, is regarded as a unit of that kind; and may have a co-efficient. Thus, in the expressions, $3 \times \overline{a - b + x}$, and $5(x + ax - y)$; $a - b + x$ is a quantity taken 3 times, and $x + ax - y$ is a quantity taken 5 times. Like quantities of this kind can be united; thus, $2(ay - bx + x) + 5(ay - bx + x) = 7(ay - bx + x)$.

§ 152. In uniting terms, great care must be taken that the literal part be entirely alike, both in signs and letters. Thus, $2bx + 3cx$, cannot be united. Neither can $3y - 2ay$; nor, $6(a + bx) + 2(ax + bx)$; nor, $3. \overline{ay - by} + 2. \overline{ay^2 - by}$; nor, $4(ax - bx) - 2(ax + bx)$; neither in any other case where there is the least difference in any part but the leading co-efficient.

EXAMPLES.

Unite the following quantities.

$$1. 3ax - 2y + 4ax - 5y + ax - 3y. \quad \text{Ans. } 8ax - 10y.$$

$$2. 3x + ay - 2x - ay + 4x + 3ay - 2x + 4ay. \quad \text{Ans. } 3x + 7ay.$$

$$3. 4ax - y + 3ay - 2 - 2ax + ay - 7y + 8 + 2ay + y. \quad \text{Ans. } 2ax + 6ay - 7y + 6.$$

$$4. ax - ay^2 - 3ay + 5ax - 2ay + 7ay - 4ax - 8ay^2. \quad \text{Ans. } 2ax + 2ay - 9ay^2.$$

Questions. Give an example in which literal quantities are not similar. What is said of quantities in a vinculum? In uniting, what particular care is necessary?

$$5. 3(a-y) + 4(a-y) + 2(a-y) + 7(a-y). \quad \text{Ans. } 16(a-y)$$

$$6. -4(a+b) + 3(a+b) - 2(a+b) + 7(a+b). \quad \text{Ans. } 4.\overline{a+b}.$$

$$7. 2(ab+x) + 3(ax+b) - 4(x-y) - 2(ab+x). \quad \text{Ans. } 3(ax+b) - 4(x-y).$$

$$8. 7y - 4(a+b) + 6y + 2y + 2(a+b) + (a+b) + y - 3(a+b). \quad \text{Ans. } 16y - 4(a+b).$$

$$9. x^2 + ax - ab + ab - x^2 + xy + ax + xy - 4ab + x^2 + x^2 - x + xy + xy + ax. \quad \text{Ans. } 2x^2 + 3ax - 4ab + 4xy - x.$$

§ 153. Sometimes the subtraction is *expressed* by enclosing the quantity to be subtracted in a parenthesis, and prefixing the sign $-$. Thus, $4a - 2x + 3ax - (4x + 3ay - 2ax)$. In such cases, when the vinculum is destroyed, the signs must be changed. Thus, $4a - 2x + 3ax - 4x - 3ay + 2ax$. But if there is a co-efficient immediately before the vinculum, the vinculum cannot be destroyed, nor the signs changed; as in $ax - 2(ab - 3x)$. Because such quantities are considered as only one unit. § 151.

$$10. x + 12 - ax + y - (48 - x - ax + 3y). \quad \text{First subtract. See page 68. Ans. } 2x - 36 - 2y.$$

$$11. ab - 4xy - a - x^2 - (2xy - b + 14x + x^2). \quad \text{Ans. } ab - 6xy - a + b - 14x - 2x^2.$$

$$12. 3(x+y) + (4.\overline{x+y}). \quad \text{Ans. } 7(x+y).$$

$$13. 2(a+b) - x - (3.\overline{a+b} - x^2). \quad [\text{See § 153.}] \quad \text{Ans. } x^2 - (\overline{a+b}) - x.$$

$$14. \text{From } 4.\overline{a+b}, \text{ take } \overline{a+b} - 3.\overline{x-y}. \quad \text{Ans. } 3.\overline{a+b} + 3.\overline{x-y}.$$

$$15. a+b - (2a - 3b) - (5a + 7b) - (-13a + 2b). \quad \text{Ans. } 7a - 5b.$$

Questions. How may subtraction be expressed? What if the vinculum in such expressions is destroyed? When cannot the vinculum be destroyed? Why?

$$16. 37a - 5x - (3a - 2b - 5c) - (6a - 4b + 3h).$$

$$\text{Ans. } 28a + 6b - 5x + 5c - 3h.$$

§ 154. After the subtraction of a quantity has been performed, we may transform the expression by changing the signs to their original form and resupplying the parenthesis. Thus, $2x - (3a + 2y - x)$, becomes, when subtracted, $2x - 3a - 2y + x$; and this latter expression may be restored back to $2x - (3a + 2y - x)$.

By the same principle, in any quantity, we may suppose there has been a subtraction, and therefore transform the expression to what it *may* have been.

| | | |
|-------------------|-------|------------------------------|
| | Thus, | $2x - 3y + 7a + ax - 1,$ |
| may be changed to | | $2x - (3y - 7a - ax + 1).$ |
| | | $ab - 3x - 4y - 2ax + 3a,$ |
| may become either | | $ab - 3x - 4y - (2ax - 3a)$ |
| or | | $ab - 3x - (4y + 2ax - 3a).$ |
| or | | $ab - (3x + 4y + 2ax - 3a).$ |

§ 155. When similar quantities have *literal* co-efficients; as, $mx + nx$, $ay^2 - by^2$, &c.; a compound quantity may be expressed by placing the co-efficients of the similar quantities one after another, (with their proper signs,) in a parenthesis; and then annexing their common letter or letters. Thus, $mx + nx$ may be expressed $(m + n)x$; $ay^2 - by^2$ by $(a - b)y^2$; $axy + byx - abxy$ by $(a + b - ab)xy$, &c.

$$17. \text{ From } ax - rx + ny - y^2, \text{ take } (a - n)y + (a + r)x - (1 + a)y^2.$$

$$\text{Ans. } 2ny - 2rx - ay + ay^2.$$

$$18. \text{ From } a(x + y) + b(x + y), \text{ take } c(x + y) - d(x - y) + x - y.$$

$$\text{Ans. } (a + b - c) \cdot (x + y) + (d - 1) \cdot (x - y).$$

Questions. When a quantity has been subtracted, how can it be added again so as to preserve all the terms? Give an example. How will this principle enable us to transform an algebraical expression? Give an example. What may be done with literal co-efficients?

§ 156. By § 154, we determined that $1-rx-tx$ is the same as $1-(rx+tx)$, which by § 155 is $1-(r+t)x$; and universally, when we wish to enclose literal co-efficients in a parenthesis; if the first of them has the sign $-$, that sign may be put *before* the parenthesis, and the signs of all the enclosed quantities changed. Thus, $1-mx+px-qx+arx$ may be written $1-(m-p+q-ar)x$.

§ 157. By § 146, $-b$ is added to a , so as to make $a-b$. Whence it will be seen that an *algebraical sum* is different from an *arithmetical sum*; and signifies, not that it is greater than one of its parts, but that it is the result of putting together two or more different values, either of the same or of a contrary signification. The same method is employed in common language, when, in order to show what a man's fortune is, we declare *what is owed to him* and *what he owes*. In both an algebraical sum, and an inventory of a man's effects, the *real positive value* may be much smaller than any one item in the account, and even *less than nothing*.

§ 158. Hence, it may be that, after uniting terms, we shall have a negative quantity, $-a$. This shows that, having a quantity to be subtracted, we did subtract all that we could, and have yet more to be subtracted if we had anything from which to subtract.

EXERCISES IN EQUATIONS.

§ 159. The learner must now generalize the *problems* in sections 3 and 4 of Equations, pages 34 and 39. This he can easily do, if he takes care to make one of the first letters of the alphabet stand for each numeral quantity; say a for the first mentioned, b for the second, &c. When the same numeral quantity occurs more than once in the same question, the same letter must stand for it each time.

Questions. How, in such cases, can we *change* the signs? How does an algebraical sum differ from an arithmetical sum? How may an algebraical sum be illustrated in common language? What does a negative quantity standing by itself denote?

II.

MULTIPLICATION OF ALGEBRAICAL QUANTITIES.

§ 160. We have shown, § 39, that a simple literal quantity may be multiplied by writing the multiplier before that quantity. This is the case whether the multiplier is numeral or literal. Thus, a times x , is written ax ; $b \times c = bc$. In the same manner, a times bc becomes abc ; and f times abc becomes $fabc$. As $f \times abc$ is the same as $abc \times f$, we see that it is of no consequence what order we make of the letters in the product. $abcd = acdb = cadb$, &c. But it is generally more convenient to follow the order of the alphabet.

CASE 1.

§ 161. Therefore, to multiply one simple quantity by another, *write the quantities one after another, without any sign between them*. Thus, abx times $cfy = abxcfy$; $5ax$ times $cdf = 5acdfx$. But, if there are more than one NUMERICAL co-efficient, those co-efficients must be multiplied as in arithmetic, and placed before the product of the literal quantities. Thus, $3a \times 2x = 3.2ax = 6ax$. $2bc \times 5rs = 10bcrs$.

EXAMPLES.

- | | |
|--------------------------------|-------------------|
| 1. Multiply a by b . | Ans. ab . |
| 2. Multiply ab by c . | Ans. abc . |
| 3. Multiply ab by cd . | Ans. $abcd$. |
| 4. Multiply $2acx$ by by . | Ans. $2abcyx$. |
| 5. Multiply $3brs$ by $2mnx$. | Ans. $6bmnr sx$. |

Questions. How is a simple literal quantity multiplied? In what order should the factors be written? What is the operation when there are numeral co-efficients?

6. Multiply $2adn$ by $5cmx$.
7. Multiply $3rsyop$ by $4antx$.
8. Multiply $2abcx$ by $8abrx$.

§ 162. By the foregoing principle, $a \times a = aa$. Now, as in algebra the same factor is often found two or more times, Stifelius (A. D. 1554) adopted a method for shortening such expressions, in which he has ever since been followed. The method is this: *when the same letter enters as a factor two or more times into any quantity, we write the factor but once, and put at the right of it and a little raised, a figure denoting how many times it has been multiplied.* Thus, aa is written a^2 ; bbb is written b^3 ; $xxxx$ is written x^4 ; $aabbbyyyy$ is written $a^2 b^3 y^4$.

§ 163. Mathematicians are accustomed to call aa or a^2 , the second power of a , or, *a-second power*; a^3 , is called *a-third power*, &c.

§ 164. The figure that denotes the power of any quantity is called the *exponent* or *index* of that quantity.

§ 165. All quantities are said to have an exponent, either expressed or understood. Thus, a is the same as a^1 ; $b = b^1$; &c. The written exponent affects no letter except the one over which it is written; unless it is denoted by a vinculum.

§ 166. Great care must be taken by the pupil not to confound the *co-efficient* with the *exponent*; as their effects are entirely different. The *co-efficient* shows *addition*, the *exponent* denotes *multiplication*. For example, if $a = 5$, then $3a = 5 + 5 + 5 = 15$; but $a^3 = 5 \times 5 \times 5 = 125$.

Questions. About what time did Stifelius write? What if two or more factors are represented by the same letter? How are such quantities read? What effect has the exponent when it is on the right of *several* letters? What if there is no exponent written over a letter? Explain the difference between an exponent and a co-efficient.

CASE 2.

§ 167. We have seen that $a^2 = aa$; and that $a^3 = aaa$; now $aa \times aaa = aaaaa$. So we see that $a^2 \times a^3 = a^5$. Hence we establish the rule that *when both multiplier and multiplicand are denoted by the same letter, their product is found by adding their exponents.* $x^3 \times x^4 = x^7$, $y^3 \times y^2 = y^5$; &c.

§ 168. In order to facilitate the practice of multiplication, it is best to observe the following method: First, determine the sign, then the co-efficient, then the letters in their order, and then the exponents.

EXAMPLES.

- | | |
|---|----------------------|
| 9. Multiply $2amn$ by a . | Ans. $2a^2mn$. |
| 10. Multiply $3abcx$ by $4ax$. | Ans. $12a^2bcx^2$. |
| 11. Multiply $5bcmn$ by $3bc$. | Ans. $15b^2c^2mn$. |
| 12. Multiply $6a^3xy$ by $4ax^2y$. | Ans. $24a^4x^3y^2$. |
| 13. Multiply $4a^2bcx$ by a^4b^2c . | Ans. $4a^6b^3c^2x$. |
| 14. Multiply $3a^3m^2$ by a^4m^3n . | |
| 15. Multiply $5a^7x^6$ by $4a^3x^4$. | |
| 16. Multiply $2m^5r^2a^2x$ by $9a^7x^4rn^2$. | |
| 17. Multiply $3b^4c^2n^5$ by $8a^3d^2mn^3$. | |
| 18. Multiply $7a^3y^2x^4$ by $12a^5b^2x$. | |
| 19. Multiply $9m^7x^2a^3$ by $7a^2b^3m$. | |
| 20. Multiply $12a^2m^6nx^5$ by $18b^3cn^8y$ | |

CASE 3.

§ 169. When one of the factors is a compound quantity, (§65 and 67,) we multiply each term by the other factor, and

Questions. How is multiplication performed when there are exponents? What is a good method in multiplication? What if one of the factors is a compound quantity?

set down their products, each with their proper sign. In doing this, we generally begin at the *left*; thus,

$$\text{Multiplied by } \frac{a+bc}{ax+bcx}$$

$$\text{Multiplied by } \frac{ab-c}{a^3b-ac}.$$

21. Multiply $ax+bx$ by xy . Ans. ax^2y+bx^2y .
22. Multiply $2a^3bc-rxy^3$ by $2ay$. Ans. $4a^3bcy-2arxy^4$.
23. Multiply $7ay+1$ by $3r$. Ans. $21ary+3r$.
24. Multiply $2xy+ab+c^2$ by $2ax$.
25. Multiply $5-7x+2a^3b$ by $4ac^2y$.
26. Multiply $3m^2n-3rp+2sx^3$ by $6a^3bd$.
27. Multiply $2en-4an+5$ by $12a^2x^5$.
28. Multiply $4y+y^2$ by $2xy$.
29. Multiply $3a-4b+4$ by $5y$.
30. Multiply $6ax^2-a^2-1$ by $3ax^2$.
31. Multiply $4+3a-x^2$ by ay .
32. Multiply $2a+5b^2+3c-5e$ by $3a^2$.
33. Multiply $7b^3-4+a^2x-x^2$ by $4a^2x$.
34. Multiply $6x+7a-axy-2y$ by $3bx^2$.

CASE 4.

§ 170. As $(a+b) \times x = ax+bx$; we know that $x \times (a+b)$ also $= ax+bx$. Whence we see that if we wish to multiply x by $a+b$; we first find the product of a times x , and then add to it the product of b times x . And generally, *when the multiplier and the multiplicand is each composed of several terms, the product is made up of the sum of the products of the multiplicand by each term of the multiplier.*

$$\text{Thus, } (x+y) \times (a+b) \\ = \left\{ \frac{x+y}{a} \right\} \text{ and } \left\{ \frac{x+y}{b} \right\} \text{ or } \left\{ \frac{x+y}{ax+ay+bx+by} \right\}.$$

Question. How is multiplication performed when there are several terms in the multiplier?

EXAMPLES.

35. Multiply $2a+3b$ by $4a+5b$.

$$\begin{array}{r} 2a+3b \\ 4a+5b \\ \hline 8a^2+12ab \\ \quad +10ab+15b^2 \\ \hline 8a^2+22ab+15b^2 \end{array}$$

Uniting terms,

36. Multiply $a+b$ by $a+b$. Ans. $a^2+2ab+b^2$.

37. Multiply $a+b-c$ by a^2+b .
Ans. $a^3+a^2b-a^2c+ab+b^2-bc$.

38. Multiply $x^2+2xy+y^2$ by $x+y$.
Ans. $x^3+3x^2y+3xy^2+y^3$.

39. Multiply $a-x$ by $2a+3x$. Ans. $2a^2+ax-3x^2$.

40. Multiply $x-4$ by $x+8$. Ans. $x^2+4x-32$.

41. Multiply $x-y$ by $x+y$. Ans. x^2-y^2 .


42. Multiply $4a^2-3xy^2+7ax-ay$ by $2a^2+x^2y$.

43. Multiply $a^2bc^3+3a^3c^2-5b^2c$ by ax^2+yx^2 .

44. Multiply $m^3a^2+2n^2b^3-1-m^4n$ by $2m^3b^2+4a^3n^4$.

45. Multiply $8a^2x^3y^4-2a^4x^3y^2+a^5y^5$ by $3a^2m+2ny^4$.

CASE 5.

§ 171. As $(a-x) \times b = ab - xb$; we also know that $b \times (a-x) = ab - xb$. Therefore, to find $a-x$ times b , we first find the product of a times b , and then subtract from it x times b . This principle is very plain in figures. Thus, suppose we have 7 times 8, and wished only 4 times 8; if we take 3 times 8 away from the 7 times 8, we should obtain 4 times 8; as $56-24=32$. Whence we have the general rule, *if there is a negative quantity in the multiplier, the product of that quantity and the multiplicand must be subtracted from the product of the positive quantities and the multiplier*.  This explains the rule in § 173.

Question. What if there is a *negative* term in the multiplier?

$$\text{Thus } (a+b) \times (c-d) \\ = \left\{ \begin{array}{c} a+b \\ c \\ \hline ac+bc \end{array} \right\} \text{ minus } \left\{ \begin{array}{c} a+b \\ d \\ \hline ad+bd \end{array} \right\} \text{ or } \frac{a+b}{c-d} \\ \hline ac+bc-ad-bd.$$

Here we see that the product of d into $a+b$ is subtracted from the product of c into $a+b$; and therefore the signs of $ad+bd$ are changed to $-ad-bd$.

§ 172. By examining the answer of this last example, we shall observe a principle which will enable us to be more rapid in the multiplying operation. It is this: *when we multiply $a + \text{term}$ by $a + \text{term}$, the product in the answer is $a + \text{term}$; and when we multiply $a + \text{term}$ by $a - \text{term}$, the product in the final answer is $a - \text{term}$.* It will be well for the pupil to explain this.

By understanding this principle, we are able to set the final answer down at first.

EXAMPLES.

46. Multiply $a+b$ by $a-b$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \quad -b^2 \end{array} \quad \text{Ans. } a^2-b^2.$$

47. Multiply $x+8z$ by $3x^2-7xz$.

$$\text{Ans. } 3x^3+17x^2z-56xz^2.$$

48. Multiply x^2+xy+y^2 by $x-y$. Ans. x^3-y^3 .

49. Multiply $2x+3y$ by $3x-4y$. Ans. $6x^2+xy-12y^2$.

50. Multiply $b^3+b^2x+bx^2+x^3$ by $b-x$. Ans. b^4-x^4 .

51. Multiply $a-b$ by $c-d$.

$$\left\{ \begin{array}{c} a-b \\ c \\ \hline ac-bc \end{array} \right\} \text{ minus } \left\{ \begin{array}{c} a-b \\ d \\ \hline ad-bd \end{array} \right\} \text{ or } \frac{a-b}{c-d} \\ \hline ac-bc-ad+bd.$$

Question. How is the subtraction performed? By what principle can we multiply and subtract at the same time?

Here we see that in subtracting d times $a-b$, we change the signs of $ad-bd$ to $-ad+bd$.

§173. Whence we learn, that when the signs are *alike*, the product is *positive*; when the signs are *unlike*, the product is *negative*; or, in abbreviated language,

- + multiplied by +, produces +
- + multiplied by -, produces -
- multiplied by +, produces -
- multiplied by -, produces +.

And by remembering this we can always set down the *final answer* at first.

52. Multiply $a-x$ by $a-x$.

$$\begin{array}{r} a-x \\ a-x \\ \hline a^2-ax \\ -ax+x^2 \\ \hline a^2-2ax+x^2. \end{array}$$

53. Multiply $2x-3a$ by $4x-5a$.

$$\text{Ans. } 8x^2-22ax+15a^2.$$

54. Multiply $2a-5y$ by $a-2y$. Ans. $2a^2-9ay+10y^2$.

55. Multiply a^2+ac-c^2 by $a-c$. Ans. $a^3-2ac^2+c^3$.

56. Multiply $a+b-d$ by $a-b$. Ans. a^2-ad-b^2+bd .

57. Multiply $4x-5a-2b$ by $3x-2a+5b$.

$$\text{Ans. } 12x^2-23ax+14bx+10a^2-21ab-10b^2.$$

58. Multiply $x^3-y^3-z^3$ by $x-y-z$.

59. Multiply $6+xy-a^2-my^3$ by $a^2-3x^3+y^4$.

60. Multiply $2a^3b-3ac^2+4b^3c^2-1$

$$\text{by } 2a^3c^2-5b^2c-8a^2.$$

61. Multiply $3a^2x^3-4x^5+2b^2c^2-abc-1$

$$\text{by } 2xb^2+6c^2x-bc-1.$$

62. Multiply $4mx-nr-2m^2r+3rx^2-1$

$$\text{by } 4m^2+3n^3-r^2-1$$

Question. What is the general rule for the signs?

§ 174. The rule for the signs which has been established for compound quantities is also extended to simple quantities. It may seem strange to the young student to hear of multiplying $+a$ by $-x$, or $-a$ by $+x$, or $-a$ by $-x$. But, as it was intimated, § 157, 158, a negative quantity implies merely a *contrary value* from a positive quantity; and therefore, though in algebra, the operation *seems* to be upon *imaginary* quantities, yet in the *application* of the operation to arithmetic and geometry, the mind readily *understands* their signification. The extension of this rule to simple quantities, will interpret the peculiar results to which algebraical operations sometimes lead.

§ 175. It is often the case that it is better to *denote* the multiplication of compound quantities, than to perform it, on account of operations that follow. Thus, $a-b$ times $x+y$, may be written $(a-b) \cdot (x+y)$. No general rule can be given to determine when one method is preferable to the other. Experience is the best teacher in this particular. But it was thought best to mention it in this place.

When, after the multiplication has been denoted, the several terms are actually multiplied, the expression is said to be *expanded*.

§ 176. In uniting the terms, if there are literal co-efficients of similar quantities, they may be enclosed in a parenthesis, as was shown in addition. § 154 and 155.

63. Multiply $mx^2 - nx - r$ by $nx - r$.

$$\text{Ans. } mnx^3 - (n^2 + mr)x^2 + r^2.$$

64. Multiply $px^2 + qx - y$ by $mx - n$.

$$\text{Ans. } mp x^3 + (mq - np)x^2 - (my + nq)x + ny.$$

65. Multiply $ax^2 - bx + c$ by $x^2 - cx + 1$.

66. Multiply $px^2 - rx + 9$ by $x^2 - rx - q$.

Questions. How may multiplication be denoted? In what cases may literal co-efficients be enclosed in a parenthesis?

III.

GENERAL PROPERTIES OF NUMBERS.

§ 177. We have before stated that algebraical operations, (§ 139,) by reason of the quantities themselves being retained in their original value, do show us, in their results, important general principles. We will here make a few multiplications of some quantities, whose results show us some remarkable general properties of numbers. These properties the pupil should remember, as they are of frequent use in the subsequent parts of this study.

§ 178. Suppose we have two numbers, a and b , of which a is the greatest. Then their sum $= a+b$; and their difference $= a-b$. Then

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ - ab - b^2 \\ \hline a^2 - b^2 \end{array}$$

By this operation, we find a general property of numbers which it would be difficult to find by any arithmetical operation. It is that, *if we multiply the sum of two numbers by their difference, the product will be the difference of the squares of those numbers.*

§ 179. Again, take the same quantities, and multiply their sum, by their sum.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ + ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

By this operation we find the following general property

Questions. How may be found the difference of the squares of two quantities? What is the square of the sum of two numbers?

The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two numbers, plus the square of the last number.

§ 180. Again, take the same quantities, and multiply their difference, by their difference.

$$\begin{array}{r}
 a - b \\
 a - b \\
 \hline
 a^2 - ab \\
 \quad - ab + b^2 \\
 \hline
 a^2 - 2ab + b^2
 \end{array}$$

Therefore, the square of the difference of two numbers, is equal to the square of the first number, minus twice the product of the two numbers, plus the square of the second.

§ 181. The only difference between the square of the sum, and the square of the difference, is in the second term; being in one, positive, and in the other, negative. Let us subtract one from the other.

$$\begin{array}{r}
 a^2 + 2ab + b^2 \\
 a^2 - 2ab + b^2 \\
 \hline
 4ab
 \end{array}$$

The actual difference between the square of the sum, and the square of the difference, is four times the product of the two numbers.

§ 182. If when the *sum* of two quantities has been raised to the second power, and the co-efficient of the second term has been rejected, the quantity thus obtained be multiplied by the difference of the two original quantities; the result will be the *difference of the third powers of the two quantities*. Also, if we perform the same operation with the *difference* of

Questions. What is the square of the *difference* of two numbers? How do the *expressions* of the foregoing squares differ? What is their *actual* difference in quantities? How may we obtain the *difference* of the *third* powers of two quantities? How may we obtain the *sum* of the *third* powers of two quantities?

the two quantities, multiplying by their sum, we shall obtain *the sum of the third powers of the two quantities.*

$$\begin{array}{r} \text{Thus,} \quad \begin{array}{r} a+b \\ a+b \\ \hline a^3+2ab+b^3 \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ \hline a^3-2ab+b^3 \end{array} \end{array}$$

Rejecting the co-efficients,

$$\begin{array}{r} \begin{array}{r} a^3+ab+b^3 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ -a^2b-ab^2-b^3 \\ \hline a^3-b^3 \end{array} \qquad \begin{array}{r} a^3-ab+b^3 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ +a^2b-ab^2+b^3 \\ \hline a^3+b^3 \end{array} \end{array}$$

FACTORS.

§ 183. In all simple terms consisting of more than one letter, the factors are evident on inspection. The same is true of polynomials that have a simple quantity for a factor. Thus, $10a^2b^2x + 15a^3bx - 20a^3b^3y$, may be decomposed into $5a^2b(2bx + 3ax - 4ab^2y)$.

§ 184. In order to ascertain the factors of any quantity, first see what is the greatest quantity that will divide every term of the given quantity, and set that down as one factor. Then divide the given quantity by the factor set down, and the quotient will be the other factor.

§ 185. By § 178, 179, $a+b$ is a factor of $a^3+2ab+b^3$; and $a-b$ is a factor of $a^3-2ab+b^3$. By § 177, either $a+b$ or $a-b$ is a factor of a^3-b^3 . By § 182, $a+b$ is a factor of a^3+b^3 , and $a-b$ is a factor of a^3-b^3 .

Any quantity is a factor of an expression made up of the same quantity and an exponent.

EXAMPLES.

Find the factors of the following.

1. $4a^3y - 4a^2x + 8a^4xy - 12a^7$.
2. $a^3b^2x^2 + 3a^2b^2x^3 - a^2x^2 + 4a^7x^3y - 5a^3x^5y^2$.
3. $2a^3bx^2 - 4ab^4x + 6a^5b^2x - 2abx$.
4. $a^3 + 2ab + b^3 + a^2 - b^2 + a^3 + b^3$.

IV.

DIVISION OF ALGEBRAICAL QUANTITIES.

§ 186. Division may be *represented* by the sign \div , as $a \div b$, is read *a, divided by b*; $(a+b) \div (c-d)$, is read *a+b, divided by c-d*. But the most usual way to denote division (§ 72) is to write the divisor underneath the dividend; thus, $\frac{a}{b}$, $\frac{a+b}{c-d}$.

§ 187. But it often happens (§ 76, 78) that the fraction made by this representation is an improper fraction; and one in which the numerator can be actually divided by the denominator. In such cases, it is generally best to perform the division. We have always done so in the former part of this treatise. Thus, $4x$ divided by 2 equals $2x$; $\frac{10x}{5} = 2x$.

§ 188. Let us first look at the case where the same quantity is in both the dividend and the divisor. $7a \div 7 = a$; $12b \div 12 = b$. In the same manner, $ab \div a = b$; $dc \div d = c$. This may easily be proved. For, ab is the product of a into b ; and of course if we divide by what was the multiplier, we shall obtain the old multiplicand again; as may be seen by trying the product of any two numbers.

§ 189. Whence we derive the general rule, that *when the divisor is found as a factor in the dividend, the division is performed by erasing that factor from the dividend*. $amn \div a = mn$, because $amn = a$ times mn . $amn \div m = an$, because $amn = m$ times an . $amn \div n = am$, because amn is n times am .

Questions. How is division usually represented? What if the same quantity is in the dividend and in the divisor? Prove it by examples.

EXAMPLES.

1. Divide $8c$ by 8 . Ans. c .
2. Divide bc by b . Ans. c .
3. Divide $7m$ by 7 . Ans. m .
4. Divide am by a . Ans. m .
5. Divide bd by b .
6. Divide $7ab$ by 7 .
7. Divide cab by c .
8. Divide bad by b .
9. As abd is the same product as the last, divide that by b . Ans. ad .
10. Divide cde by c .
11. Divide the same product in another form, thus, dec by d . Ans. ec .
12. Divide abc by c .
13. Divide abc by b .

§ 190. As $a^2 = aa$, it is evident that if we divide a^2 by a , the quotient is a ; because we take away one of the written a 's. So, if we divide a^5 by a , the quotient is a^4 ; because a^5 is the same as $aaaaa$, which $\div a$, gives $aaaa$ or a^4 . So, if we divide a^5 by a^2 , the quotient is aaa or a^3 . And if we divide a^5 by a^4 , the quotient is a ; because $aaaaa \div aaaa = a$. Hence we see, that when *there are exponents in either the dividend or divisor, the division is performed by subtracting the exponent of the divisor from the exponent of the dividend.* $b^5 \div b^2 = b^3$; $x^7 \div x^3 = x^4$.

§ 191. As every literal quantity is understood to have the number 1 for its co-efficient, it is evident that if we divide by 1, the quotient would be the same literal quantity. Thus, $a \div 1 = a$. And again, if we divide a literal quantity by

Questions. How is division performed when there are exponents? Why? How when the divisor is the only quantity in the dividend? Why?

itself, the quotient will be 1. Thus, $1a \div a = 1$: the divisor being a factor in the dividend.

§ 192. It sometimes happens that the co-efficient contains the divisor as a factor. Thus, $8a$ is the same as 2 times $4a$, or 4 times $2a$; and therefore can be divided by 2 or by 4. In the same manner $8ab$ may be divided by $2a$, or $4a$, or $2b$, or $4b$; because it is $2a$ times $4b$, or $4a$ times $2b$.

We have only to remember to take those factors out of the dividend which are equal to the divisor. And in general, *when there are co-efficients in both the divisor and the dividend, divide the co-efficient of the dividend by the co-efficient of the divisor*; and then proceed with the literal quantities as before directed. $10abc \div 5b = 2ac$; $12a^3xy \div 3a^2y = 4ax$.

EXAMPLES.

- | | |
|---|--------------------|
| 14. Divide a^3 by a^2 . | Ans. a . |
| 15. Divide x^6 by x^2 . | Ans. x^4 . |
| 16. Divide a^3b^4y by a^2b . | Ans. ab^3y . |
| 17. Divide $d^3c^4x^7$ by dc^2x^4 . | Ans. $d^2c^2x^3$. |
| 18. Divide a^4m^5x by a^4m^3 . | |
| 19. Divide $a^4x^6y^7$ by ax^4y^3 . | |
| 20. Divide d^6y^7 by dy . | |
| 21. Divide $p^3r^7s^2t$ by r^5st . | |
| 22. Divide $ab^3c^4d^5$ by ab^2cd^5 . | |
| 23. Divide ax^7y^8 by ax^3y^5 . | |
| 24. Divide $c^5r^7s^3tx^4y^7$ by $c^2r^5ty^3$. | |
| 25. Divide $6a^2bc^7$ by $2bc^5$. | Ans. $3a^2c^2$. |
| 26. Divide $12ax^2y^4$ by $4ay^2$. | Ans. $3x^2y^2$. |
| 27. Divide $21bc^3xy^6$ by $3cy^2$. | |
| 28. Divide $42c^7d^8x^3$ by $6c^2d^5x$. | |

Questions. How is division performed when there are co-efficients? Explain.

29. Divide $36pr^3st^4$ by $4rst^3$.
30. Divide $54m^5n^2x^3y$ by $9m^3ny$.
31. Divide $66a^2c^5dx^6$ by $3a^2c^5x^3$.
32. Divide $48c^2r^7x^3y^5$ by $8cr^5xy^2$.
33. Divide $72a^3r^3m^4$ by $18a^2rm^3$.

CASE 2.

§ 193. We have shown, § 169, that $(a+b) \times c = ac + bc$. Of course $(ac+bc) \div c = a+b$; where we see that *when we divide a compound quantity by a simple quantity, we divide each of the terms by that quantity.*

§ 194. We must also recollect that, as $+$ multiplied by $+$, makes $+$ in the product, so $+$ in the product divided by $+$, must make $+$ in the quotient. And that as $+$ multiplied by $-$ makes $-$, so $-$ in the product divided by $-$, will bring back the $+$ in the quotient. So that *when the signs are alike in the dividend and divisor, the sign in the quotient is $+$.* Thus, $a \times b = ab$; both of which are $+$. Also $-a \times -b = ab$; and $-ab \div -b = +a$.

§ 195. Again, as $-$ multiplied by $+$ makes $-$, so in the product, $-$ divided by $+$, brings back $-$ in the quotient. Also, $-$ multiplied by $-$ makes $+$; and of course, $+$ in the product divided by $-$, brings $-$ in the quotient. That is, *when the signs in the divisor and dividend are unlike, the sign in the quotient is $-$.*

EXAMPLES.

- | | |
|--|---------------------|
| 34. Divide $2ad+8a^2c$ by $2a$. | Ans. $d+4ac$. |
| 35. Divide $8d^3m^2-12d^5m^3$ by $4dm^2$. | Ans. $2d^2-3d^3m$. |
| 36. Divide $4xy+6x^2$ by $2x$. | Ans. $2y+3x$. |
| 37. Divide $abc-acd$ by ac . | Ans. $b-d$. |
| 38. Divide $12ax-8ab$ by $-4a$. | Ans. $-3x+2b$. |

Questions. How is a compound quantity divided? What are the rules for the signs? For what reason?

39. Divide $10xz + 15xy$ by $5x$.
40. Divide $15ax - 27x$ by $3x$.
41. Divide $18x^2 - 9x$ by $9x$.
42. Divide $abc - bcd - bcx$ by $-bc$.
43. Divide $3x + 6x^2 + 3ax - 15x$ by $3x$.
44. Divide $3abc + 12abx - 9a^2b$ by $3ab$.
45. Divide $40a^3b^2 + 60a^2b^2 - 17ab$ by ab .
46. Divide $15a^2bc - 10acx^2 + 5ad^2c$ by $-5ac$.
47. Divide $20ax + 15ax^2 + 10ax - 5a$ by $5a$.

§ 196. It is evident that we may divide by either factor. Thus, $ax + bx$ may be divided by x , and the quotient will be $a + b$; or it may be divided by $a + b$, and the quotient will be x . This may appear singular to the young pupil; but he is to recollect that division is merely separating the dividend into factors, being careful to make one of them of a given magnitude; that is, to make it the same as the given divisor. For illustration, $ax + bx$ means that x is taken a times, and also b times. Therefore it is taken $(a + b)$ times; and in the whole quantity, $a + b$ is the co-efficient of x , so that $(ax + bx) = (a + b)x$.

§ 197. Now we know that x times $a + b = ax + bx$; and also that $a + b$ times $x = ax + bx$. Whence, the product $(ax + bx) \div (a + b) = x$. Therefore we conclude that *if the divisor contains just as many terms as the dividend, with corresponding signs; and the first term of it is a factor in the first term of the dividend, the second term of it in the second of the dividend, and so on through each of them respectively; and the remaining factor in every term of the dividend being the same; that remaining factor only shall be the quotient.*

Questions. What is the rule for dividing when the number of terms in the dividend and divisor is the same? Explain and illustrate!

EXAMPLES.

48. Divide $ax+bx+cx$ by $a+b+c$. Ans. x .

49. Divide $bac+bc^2x-bx^3$ by $ac+c^2x-x^3$. Ans. b .

50. Divide $c^2ax-2abx-3xy+x$ by $ac^2-2ab-3y+1$.

51. Divide $cd^2x-abd^2+d^2x^3-l^2$ by $cx-ab+x^3-1$.

52. Divide $a^2y-bcy+xy$ by a^2-bc+x .

53. Divide $6ahm-14abm-3cdm$ by $6ah-14ab-3cd$.

§ 198. If the letters of the divisor are not found in the dividend, the division is expressed, as we have before shown, § 186, by writing the divisor underneath the dividend, in the form of a vulgar fraction.

EXAMPLES.

54. Divide $4y+7x$ by $a-b$. Ans. $\frac{4y+7x}{a-b}$.

55. Divide $3a+2b^2-c$ by $a+c$.

56. Divide $a^3-x^2b+c^5$ by a^3-b^2 .

57. Divide $3a^2c+2b^3+c$ by $2c$.

§ 199. When the dividend is a compound quantity, the divisor may be placed underneath the *whole* dividend if we choose. It may also be placed under *each term* of the dividend, which is the same as dividing each term, according to § 193. By this method, the answer of the last sum would be

$$\frac{3ac^2}{2c} + \frac{2b^3}{2c} + \frac{c}{2c}. \quad \text{Answer the following by both methods.}$$

58. Divide $3ab+b+2ab$ by a . Ans. $\frac{3b}{a} + \frac{b}{a} + \frac{2ab}{a}$.

59. Divide $6a+ab-3b$ by $2b$.

60. Divide $2x+2y+3ax-2a^2y$ by $3ay$.

61. Divide $ax^2-bx-a^2b^2+ab$ by $2b$.

Questions. What is the operation when the same quantities are not in both the dividend and divisor? Must the divisor always be put under the whole dividend?

§ 200. When we divide each term separately, we may use both methods of division; that is, we may actually *divide* such terms as we can, by § 189; and merely *express* the division in such terms as cannot be divided.

EXAMPLES.

62. Divide $cd - ax + ac + bc$ by c . Ans. $d - \frac{ax}{c} + a + b$

63. Divide $ax + bx - 2ab + 2x$ by x . Ans. $a + b - \frac{2ab}{x} + 2$.

64. Divide $2am - 3a^2b + b^2m - 3a^2m$ by $-a$.
Ans. $-2m + 3ab - \frac{b^2m}{a} + 3am$.

65. Divide $2b^2 - a^2b + 3b^2c + a^2b^2 - ac$ by b^2 .

66. Divide $ay^3 - by^2 + 4a^2b^2 - 5a^2by^2 + aby$ by ay .

67. Divide $abx^3 + a^2by - 3ab^2x + ax^2 - 7a^2y$ by ab .

68. Divide $2abm + 6a^2b + 5b^2m - 4a^2m$ by ab .

69. Divide $by - a^2b^2y + ay - aby + b^2y$ by $-by$.

70. Divide $ax - bx^2 + xy^2 + by - ay$ by $-y$.

EXERCISES IN EQUATIONS.

§ 201. The learner must now generalize the problems in section 5 of Equations, page 44. And with this section, we will make our calculations more purely algebraical than in the preceding sections; as we shall take care to use *no numeral quantities at all in stating the questions*.

§ 202. It is customary to represent those numbers which stand for *times*, by the letters m, n, p, q , &c.

1. Two persons, A and B, lay out equal sums of money in trade. A gains c (\$126,) and B loses d (\$87;) and now A's money is m (two) times as much as B's. What did each lay out? See page 44.

Question. When may we use both methods of dividing?

Let x = what each lay out.

$x + c$ = A's sum now.

$x - d$ = B's sum now.

$mx - md$ = m times B's.

Forming the equation,

$$mx - md = x + c$$

Transposing,

$$mx - x = c + md$$

Dividing by $m - 1$,

$$x = \frac{c + md}{m - 1}.$$

Substituting numbers }
for letters, }

$$\frac{c + md}{m - 1} = \frac{126 + 2 \times 87}{2 - 1} = 300.$$

In sums of this kind, the only difficulty is to determine what quantity we must divide by in the last step, to leave x alone. But this difficulty is easily overcome, by dividing mentally the left hand member by x , and observing the quotient. Of course, dividing the same member by that *quotient* will produce x ; which is our only object.

2. The 2d sum on page 45, is performed as follows :

x = the wife's age.

mx = the man's age.

$x + a$ = wife's after a years.

$mx + a$ = man's after a years.

$nx + na$ = n times the wife's age.

Forming the equation,

$$mx + a = nx + na$$

Transposing terms,

$$mx - nx = na - a$$

Dividing by $m - n$,

$$x = \frac{na - a}{m - n}.$$

Substituting numbers,

$x = 15$, the age of his wife.

All the other questions in section 5th are to be performed in the foregoing manner.

DIVISION BY COMPOUND DIVISORS.

§ 203. By examining any sum in multiplication, it will be seen that any one term in the product contains both one term of the multiplicand, and one term of the multiplier. See each term in the answer of the following example.

$$\begin{array}{r} x+y \\ a+b \\ \hline ax+ay+bx+by. \end{array}$$

Therefore, if we find any term of the product that contains the first term of the multiplier, we know that the remaining part of it is a term in the multiplicand. And, as either term of the multiplicand may be put as the first, (for $y+x$ is the same as $x+y$), we may suppose that the term which we have found by this process is the first. Thus, in the answer above, ax and ay contain a of the multiplier; and therefore either x or y , whichever we choose, is the first term of the multiplicand.

Now, by change of names, $ax+ay+bx+by$ may be the dividend, and $a+b$ the divisor, for the purpose of finding the quotient. We see that a , the first term of the divisor, is found in ax and in ay ; and we may therefore conclude that the first term in the quotient is either x or y , which are the remaining parts of those terms. Likewise, $(a+b) \times x$ or y , will be a partial dividend.

Ex. 1. Suppose it were required to divide $ac+bc+ad+bd$ by $a+b$. We see that the first term in the divisor is contained in the first term of the dividend; and therefore judge that the other factor, which is c , must be the first term in the quotient; and that c times $a+b$, which is $ac+bc$, is one of the partial

Questions. What may be said of each term in a product? What then do we know of any term in the product that contains the first term of the multiplier? What is the change of names for division instead of multiplication? Explain the first example. To what is it similar?

dividends. Subtracting $ac+bc$ from the whole dividend, and there remains $ad+bd$ still to be divided by $a+b$. The work is as follows :

$$\begin{array}{r} a+b) ac+bc+ad+bd \quad (c+d \\ \underline{ac+bc} \\ ad+bd \end{array}$$

In the last partial dividend, we see by the first term that the divisor is contained in it d times.

§ 204. This is the same method that is followed in arithmetic ; where, in order to find a quotient figure, we see how many times the first figure of the divisor is contained in the first figure of the partial dividend ; and *supposing* the number to be the true quotient, we multiply the divisor by it. But there is this difference between algebraic and arithmetical division, that in algebra the quotient is *always* the *true* one ; and also where the divisor is not found as a factor of one of the terms of the dividend, the operation of division is impossible.


Ex. 2. Divide $5a^2x+x^3+a^3+5ax^2$ by $a+x$.

§ 205. From the last example, it seems that it is best to put first in the dividend *those terms* that contain the letter which is in the *first term* of the divisor. It has also been found *easier* in practice to arrange the *powers* of this letter both in the divisor and dividend, so that the *highest* should stand *first*, the next highest next, and so on. We will adopt this plan in the present example.

$$a+x) a^3+5a^2x+5ax^2+x^3 \quad (a^2+4ax+x^2$$

In subtracting, a^3+a^2x

$$\begin{array}{r} \text{change the signs} \quad 4a^2x+5ax^2+x^3 \\ \text{mentally, and} \quad \underline{4a^2x+4ax^2} \\ \text{unite at the} \quad \quad \quad ax^2+x^3 \\ \text{same time.} \quad \quad \quad \underline{ax^2+x^3} \end{array}$$

 We find the terms of quotient by § 204. Thus,

The first term of the divisor is contained in a^3 , a^2 times ;

Questions. To what is this method similar ? What arrangement of terms is recommended ? Explain the second example.

wherefore we determine that a^3 is one term in the quotient. The product of that term and the divisor is $a^3 + a^2x$, which we subtract from the dividend, and there is left $4a^2x + 5ax^2 + x^3$. The first term of the divisor is contained in the first term of this partial dividend $4ax$ times; wherefore we determine, that $4ax$ is one term of the quotient.

We will incorporate the foregoing remarks into the following general rule.

RULE FOR DIVIDING WITH COMPOUND DIVISORS.

§ 206. *Arrange the terms of both dividend and divisor according to the powers of the same letter, beginning with the highest. Divide the first term of the dividend by the first term of the divisor, and place the result with its proper sign for the first term in the quotient.*

§ 207. *Multiply the whole divisor by this term of the quotient, setting the product under the dividend; subtract this product from the dividend, and use the remainder as the next partial dividend.*

§ 208. *Divide the first term of the partial dividend by the first term in the divisor, and place the result for the next term in the quotient. Multiply the divisor by the term last written in the quotient, subtract the product as before, and proceed in this manner as long as the first term of the dividend can be divided by the first term in the divisor.*

§ 209. *If there is a remainder in which there is no term that contains the first term of the divisor, that remainder must be made the numerator of a fraction whose denominator is the divisor; and the fraction must be put at the end of the quotient by addition.*

Questions. What is the first step in the rule? What is done with the first term of the quotient? How do we proceed with a partial dividend? What is done with the final remainder?

EXAMPLES.

3. Divide $a^3+3a^2x+3ax^2+x^3$ by $a+x$.

$$\begin{array}{r}
 a+x) a^3+3a^2x+3ax^2+x^3 (a^2+2ax+x^2 \\
 \underline{a^3+a^2x} \\
 2a^2x+3ax^2+x^3 \\
 \underline{2a^2x+2ax^2} \\
 ax^2+x^3 \\
 \underline{ax^2+x^3} \\
 0
 \end{array}$$

The first term of the divisor is contained in the first term of the dividend a^2 times.

The whole divisor

multiplied by a^2 , equals a^3+a^2x ; which subtracted from the dividend leaves $2a^2x+3ax^2+x^3$. The first term of the divisor is contained in the first term of the remainder, $2ax$ times. The whole divisor multiplied by $2ax$, equals, &c.

§ 210. It may be well to mention here, that if the signs of the divisor are changed, we shall obtain the same quantities for the quotient as before, with the exception that *their* signs will be changed also. The pupil may divide this last sum by $-a-x$; and he will obtain $-a^2-2ax-x^2$.

4. Divide x^3-6x-9 by $x-3$. For subtracting see Ex. 2.

$$\begin{array}{r}
 x-3) x^3-6x-9 (x^2+3x+3 \\
 \underline{x^3-3x^2} \\
 +3x^2-6x-9 \\
 \underline{3x^2-9x} \\
 3x-9 \\
 \underline{3x-9} \\
 0
 \end{array}$$

5. Divide $a^3+4ax^2-3a^2x-2x^3$ by $a^2-2ax+x^2$.

$$\begin{array}{r}
 a^2-2ax+x^2) a^3-3a^2x+4ax^2-2x^3 (a-x+\frac{ax^2-x^3}{a^2-2ax+x^2} \\
 \underline{a^3-2a^2x+ax^2} \\
 -a^2x+3ax^2-2x^3 \\
 \underline{-a^2x+2ax^2-x^3} \\
 ax^2-x^3
 \end{array}$$

6. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.

Ans. $a^2-2ay+y^2$

7. Divide $b^3-10b^2+33b-36$ by $b-4$. Ans. b^2-6b+9 .

8. Divide $6a^4-96$ by $3a-6$. Ans. $2a^3+4a^2+8a+16$.

9. Divide $x^3 - 3x^2b + 3xb^2 - b^3$ by $x - b$.
 Ans. $x^2 - 2xb + b^2$.
10. Divide $x^4 - y^4$ by $x - y$. Ans. $x^3 + x^2y + xy^2 + y^3$.
11. Divide $a^4 - 2x^4$ by $a + x$.
 Ans. $a^3 - a^2x + ax^2 - x^3 - \frac{x^4}{a+x}$.
12. Divide $a^3 - x^3$ by $a + x$. Ans. $a^2 - ax + x^2 - \frac{2x^3}{a+x}$.
13. Divide $x^4 + y^4$ by $x + y$.
 Ans. $x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x+y}$.
14. Divide $2y^3 - 19y^2 + 26y - 16$ by $y - 8$.
 Ans. $2y^2 - 3y + 2$.
15. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
 Ans. $24x^2 - 2ax - 35a^2$.
16. Divide $b^4 - 3y^4$ by $b - y$.
 Ans. $b^3 + b^2y + by^2 + y^3 - \frac{2y^4}{b-y}$.
17. Divide $2a^4 - 13a^3b + 31a^2b^2 - 38ab^3 + 24b^4$
 by $2a^2 - 3ab + 4b^2$. Ans. $a^2 - 5ab + 6b^2$.
18. Divide $x^3 + px + q$ by $x + a$.
 Ans. $x + p - a + \frac{q - pa + a^3}{x + a}$.
19. Divide $6x^4 + 9x^3 - 20x$ by $3x^2 - 3x$.
 Ans. $2x^2 + 2x + 5 - \frac{5x}{3x^2 - 3x}$.
20. Divide $9x^6 - 46x^5 + 95x^4 + 150x$ by $x^2 - 4x - 5$.
 Ans. $9x^4 - 10x^3 + 5x^2 - 30x$.
21. Divide $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$
 by $\frac{1}{2}x^2 - x + 3$. Ans. $\frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$.

§ 211. It sometimes happens, that in the dividend there are several terms which have the same exponent. Of that letter by which the quantity has been arranged. In such cases, they should be put directly under one another; and when they become the first of a partial dividend, there should be a quotient for each term before the subtraction is performed; as follows:

22. Divide $10a^3 + 11a^2b + 3ab^2 - 15a^2c - 5b^3c - 19abc + 15bc^2$ by $5a^2 + 3ab - 5bc$.

$$\begin{array}{r}
 5a^2 + 3ab - 5bc \left\{ \begin{array}{l} 10a^3 + 11a^2b + 3ab^2 - 5b^3c + 15bc^2 \\ -15a^2c - 19abc \end{array} \right\} 2a + b - 3c \\
 \hline
 10a^3 + 6a^2b - 10abc \\
 \hline
 \left\{ \begin{array}{l} + 5a^2b + 3ab^2 - 5b^3c + 15bc^2 \\ -15a^2c - 9abc \end{array} \right\} \\
 \hline
 \text{Product by } +b = 5a^2b + 3ab^2 - 5b^3c \\
 \text{Prod. by } -3c = -15a^2c - 9abc + 15bc^2 \\
 \hline
 0
 \end{array}$$

23. Divide $a^4 + 4a^3b + 8b^4$ by $a + 2b$.

$$\text{Ans. } a^3 - 2a^2b + 4ab + 4ab^2 - 8b^2 - 8b^3 + \frac{16b^3 + 24b^4}{a + 2b}.$$

24. Divide $64a^2 + 64ab + 16b^2 - 9d^2 - 48d - 64$ by $8a + 4b + 3d + 8$. Ans. $8a + 4b - 3d - 8$.

25. Divide $18a^2 + 33ab + 42ac - 12ad - 30b^2 + 124bc + 8bd - 16c^2 - 32cd$ by $6a + 15b - 2c - 4d$. Ans. $3a - 2b + 8c$.

26. Divide 1 by $1 - x$.

$$\begin{array}{r}
 1-x) 1 \quad (1 + x + x^2 + x^3 + x^4, \&c. \\
 \underline{1-x} \\
 x \\
 \underline{x-x^2} \\
 x^2 \\
 \underline{x^2-x^3} \\
 x^3 \\
 \underline{x^3-x^4} \\
 x^4
 \end{array}$$

27. Divide a^3 by $1 - x^3$. Ans. $a^3 + a^2x^3 + a^2x^6 + \frac{a^2x^9}{1-x^3}$.

§ 212. In the last two examples, the division may be carried on forever, like a decimal fraction. Any algebraical fraction also may be expanded by actually performing the division in this manner. But as in this example, so in many others, a few leading terms of the quotient will be sufficient to indicate the rest, without continuing the operation.

28. Expand the fraction $\frac{a}{a+x}$.

$$a+x) a \quad (1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4}, \text{ \&c.}$$

$$\begin{array}{r} a+x \\ -x \end{array}$$

$$\begin{array}{r} -x - \frac{x^2}{a} \\ \hline \end{array}$$

$$\begin{array}{r} \frac{x^2}{a} \end{array}$$

$$\begin{array}{r} \frac{x^2}{a} + \frac{x^3}{a^2} \\ \hline \end{array}$$

$$\begin{array}{r} -\frac{x^3}{a^2} \end{array}$$

$$\text{For } -\frac{x}{1} \div a = -\frac{x}{a}$$

$$\frac{x^2}{a} \div a = \frac{x^2}{a^2}$$

$$-\frac{x^3}{a^2} \div a = -\frac{x^3}{a^3}$$

It is obvious that if the division is continued, the remaining terms will be alternately $-$ and $+$; and that they will increase one power in every successive term.

§ 213. Such operations may be carried on infinitely, so as to bring the result nearer and nearer to the true quotient. And on this account mathematicians have called such expressions *Infinite series*. It must be understood, however, that it is not the *quantity* which is infinite, but the number of terms. Thus, $.666666666666, \text{ \&c.}, = \frac{2}{3}$.

$$1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \frac{x^5}{a^5} + \frac{x^6}{a^6}, \text{ \&c.}, = \frac{a}{a+x}.$$

29. Reduce $\frac{1}{1+a}$ to an infinite series.

$$\text{Ans. } 1 - a + a^2 - a^3 + a^4 - a^5 + a^6, \text{ \&c.}$$

30. Reduce $\frac{1+a}{1-a}$ to an infinite series.

$$\text{Ans. } 1 + 2a + 2a^2 + 2a^3 + 2a^4 + 2a^5, \text{ \&c.}$$

Questions. What may be done with an algebraical fraction? Is it always necessary to continue the operation? How far may the operation be continued? What name has been given to the answer? Is the quantity infinite?

V.

FRACTIONS.

REDUCTION OF FRACTIONS TO LOWER TERMS.

§ 214. We showed in § 88, that a fraction may be reduced to lower terms, without any alteration in its value, by simply dividing both terms by a number that will divide each without a remainder. Fractions that are expressed by literal quantities may frequently be reduced in the same manner. Thus, in the fraction $\frac{axy}{ab}$, both the terms may be divided by a ; and the fraction will then become $\frac{xy}{b}$.

EXAMPLES.

1. Reduce to the lowest terms the fraction $\frac{4abcx}{6adcy}$.

$$\text{Ans. Both terms } \left\{ \frac{4abcx}{6adcy} \div 2ac = \frac{2bx}{3dy} \right.$$

2. Reduce $\frac{a^2m^3y}{a^3bm^2x}$ to its lowest terms.

$$\text{Ans. } \frac{my}{abx}.$$

3. Reduce $\frac{56x^4y^6}{-7x^5y^4}$ to its lowest terms.


$$\text{Ans. } -\frac{8y^2}{x}.$$

4. Reduce $\frac{-4x^5y^2z}{5x^4y^4}$ to its lowest terms.

$$\text{Ans. } -\frac{4x^1z}{5y^2}.$$

5. Reduce $\frac{-12x^4yz}{-4x^3yz^4}$ to its lowest terms.

$$\text{Ans. } \frac{3x}{z^3}.$$

 These examples will remind the pupil, that, (because fractions are merely *expressions of division*,) when each

Questions. How may fractions be reduced to lower terms? Where must a sign be put to affect the whole fraction? What sign in each term will make *plus* for the whole? What sign in each term will make *minus* for the whole? § 195.

underneath the dividend, so as to make a fraction, and then reduce that fraction to its lowest terms.

EXAMPLES.

16. Divide $x^2 - 2xy + xy^2$ by $4xy$. Ans. $\frac{x - 2y + y^2}{4y}$.

17. Divide $10xy - 20x - 5y$ by $-5x$. Ans. $-\frac{2xy - 4x - y}{x}$.

18. Divide $7abx - 56a^2xy + 14ax^3$ by $28a^3bx^2y$.

19. Divide $8amy^3 + 16a^3xy - 24aby^3$ by $48a^2y^3 - 72ay$.

20. Divide $35a^3bc - 14ax + 42a^3$ by $21a^6 - 28a^5x + 7a^4x^2$.

21. Divide $32x^3y + 16x^2y^2 + 8xy^3$ by $24ax + 48a^2x^2$.

22. Divide $54a^2b^2 + 45a^3b^2 - 27a^2b^3$ by $24a^5b^2 + 30a^3b^5$.

23. Divide $48x^2y^3 + 12axy - 16ax$ by $4x^2y^3 + 6ax - 8xy$.

24. Divide $28(a - b + x)$ by $4m(a - b + x)$. Ans. $\frac{7}{m}$.

25. Divide $12cd(m - n)$ by $14ac(m - n)$.

26. Divide $6ah(r + p)$ by $24a(r + p)$.

27. Divide $(a + b)(m + n)$ by $(a - x)(m + n)$.

28. Divide $32abc$ by $32abcx - 32aby$.

29. Reduce the fractions $\frac{7ay}{3mn}$; $\frac{4a^3b}{28a^2y}$; $\frac{25xy^3}{12a^3b^3}$; $\frac{25xy^3}{5xy}$; $\frac{5ao}{5ab}$;

$\frac{12m^2n - 12mn^2}{12m^2n - 12mn^2}$.

§ 217. It must be remembered that when all the factors in the numerator are contained in the denominator, the answer will contain 1 in the numerator. Thus, $\frac{3a}{3ax} = \frac{1}{x}$. The same principle will also apply to the denominator.

Thus, $\frac{4ax}{4a} = \frac{x}{1} = x$.

Question. What, if in reduction, the common divisor is the same as one of the terms of the fraction?

VI. MULTIPLICATION WHERE ONE FACTOR IS A FRACTION.

§ 218. This is done, (as shown § 74, 75,) by multiplying the whole number and the numerator of the fraction together, and dividing by the denominator.

$$\text{Thus, } 2a \times \frac{3b-x}{y} = \frac{6ab-2ax}{y}; \quad \frac{a}{4}b = \frac{ab}{4}.$$

EXAMPLES.

$$1. \text{ Multiply } \frac{xy}{3a} \text{ by } 3a. \quad \text{Ans. } \frac{3axy}{3a} = xy.$$

$$2. \text{ Multiply } \frac{3z^2}{8ax} \text{ by } 8a. \quad \text{Ans. } \frac{3z^2}{x}.$$

$$3. \text{ Multiply } 3ax \text{ into } -\frac{ab}{12ax}. \quad \text{Ans. } -\frac{ab}{4}.$$

$$4. \text{ Multiply } 2ab-3xy \text{ by } \frac{2by}{3ax}. \quad \text{Ans. } \frac{4ab^2y-6bxy^2}{3ax}.$$

$$5. \text{ Multiply } \frac{aby}{cx} \text{ by } 2ax-3xy. \quad \text{Ans. } \frac{2a^2by-3aby^2}{c}.$$

$$6. \text{ Multiply } 3am^2-4x \text{ by } \frac{3x^2+a}{2m}.$$

$$7. \text{ Multiply } \frac{5ast-2m}{4ax} \text{ by } am-2a^2.$$

$$8. \text{ Multiply } 3a^2y+6x^2 \text{ by } \frac{2ab+cd}{3ax-bd}.$$

$$9. \text{ Multiply } \frac{2ar^2}{3ax-bx} \text{ by } 2ax+3x^2.$$

$$10. \text{ Multiply } 14abc-3cdx \text{ by } \frac{2am}{5mx-2cm}.$$

Questions. How is multiplication performed when one factor is a fraction? What is a factor? What if the whole number is a factor in the denominator of the fraction?

11. Multiply $\frac{10ax-6xy}{14am}$ by $3mx-2am$

12. Multiply $\frac{a}{b}$ by b .

Ans. $\frac{ab}{b} = a$.

13. Multiply $\frac{ab}{cd}$ by c .

Ans. $\frac{abc}{cd} = \frac{ab}{d}$.

§ 219. In the last example, we first multiplied the numerator by c , and then divided both the numerator and denominator by c . Now, multiplying the numerator by c and then dividing it by c , is altogether useless; because the numerator is left as it was at first. We will use then only one part of the operation; that is, dividing the denominator by c . And in general, *when the multiplier is a factor in the denominator, the multiplication is performed by canceling that factor.*

EXAMPLES.

14. Multiply $\frac{amx}{rs}$ by r .

Ans. $\frac{amx}{s}$.

15. Multiply x^3 by $\frac{abc}{x^3y}$.

Ans. $\frac{abc}{xy}$.

16. Multiply $\frac{ax^3y}{2cdn}$ by $2cn$.

Ans. $\frac{ax^3y}{d}$.

17. Multiply $2d$ by $\frac{3ars}{2cdn}$.

18. Multiply $\frac{ab+ax}{2bx}$ by $2x$.

19. Multiply $3bx$ by $\frac{ax^3+2a}{3abx}$.

20. Multiply $\frac{am^3}{12ax^3-18bx^3}$ by $6x^3$.

21. Multiply $3a$ by $\frac{6am-4x}{3am+6ax-12a^3}$.

22. Multiply $\frac{abc-bx+cx}{9axy+12a^3x-6axc}$ by $3ax$.

23. Multiply $4ab$ by $\frac{am+4ab-2m}{8abm+16ab-4abx}$.

24. Multiply $\frac{3a-b}{5x+ay}$ by $5x-b$.

VII. REDUCTION OF COMPLEX FRACTIONS TO SIMPLE ONES.

§ 220. We have shown, that when we multiply a fraction by its denominator, we obtain for the answer the same quantity as the numerator. We have also shown that where both terms of a fraction are multiplied by the same quantity, the *value* of the fraction is not altered. By these two principles, we obtain the following *rule* for reducing a complex fraction to a simple one.

§ 221. *Multiply both terms of the fraction by the denominator that is found either in the entire numerator or denominator. If the fraction is still complex, multiply the result in both terms by the remaining denominator that is found in the entire term.* Thus,

$$\frac{a+\frac{b}{c}}{1} = \frac{ca+b}{cd}; \quad \frac{a-\frac{b}{c}}{\frac{3}{4}+a} = \frac{ca-b}{\frac{3c}{4}+ca} = \frac{4ac-4b}{3c+4ac}.$$

EXAMPLES.

1. Reduce $\frac{a}{a-\frac{b}{c}}$ to a simple fraction.

Ans. Multiplying both terms by c , $\frac{ac}{ac-b}$.

2. Reduce $\frac{a-\frac{x}{y}}{ax}$ to a simple fraction.


Ans. $\frac{ay-x}{axy}$.

Questions. How may a complex fraction be changed to a simple fraction? Explain why. How may fractions be transferred from the numerator to the denominator?

EXERCISES IN EQUATIONS.

§ 223. Generalize the questions in section 6th, page 52.

1. In an orchard, $\frac{1}{m}$ ($\frac{1}{4}$) of the trees bear apples; $\frac{1}{n}$ ($\frac{1}{5}$) of them bear pears; $\frac{p}{r}$ ($\frac{2}{11}$) of them, plums; and a (81) bear cherries. How many trees are there in the orchard?

 We rarely represent *unity* by a letter; but generally use its own character, 1.

Let x = number of trees.

$$\frac{x}{m} = \text{apple-trees.}$$

$$\frac{x}{n} = \text{pear-trees.}$$

$$\frac{px}{r} = \text{plum-trees.}$$

Then,

$$x = \frac{x}{m} + \frac{x}{n} + \frac{px}{r} + a.$$

§ 224. As the equation is to be multiplied by m , and then by n , and then by r ; it is plain we may multiply it at once by their product, mnr . Of course, in multiplying a fraction, its denominator will not appear in the product. § 219.

Multiplying by mnr , $mnrx = nrx + mrx + mnp x + mnra$

Transposing, $mnrx - nrx - mrx - mnp x = mnra$

Dividing by $mnr - nr - mr - mnp$,

$$x = \frac{mnra}{mnr - nr - mr - mnp}$$

Substitute figures for letters, and find the answer.

$$\begin{aligned} x &= \frac{4 \times 5 \times 11 \times 81}{4 \times 5 \times 11 - 5 \times 11 - 4 \times 11 - 4 \times 5 \times 2} \\ &= \frac{17820}{220 - 55 - 44 - 40} \\ &= \frac{17820}{81} = 220. \end{aligned}$$

2d sum, page 53. In a certain school, $\frac{1}{m}$ of the boys learn mathematics; $\frac{p}{n}$ of them study Latin and Greek; and a study grammar. What is the whole number of scholars?

The equation is,
$$x = \frac{x}{m} + \frac{px}{n} + a.$$

Multiplying by mn ,
$$mnx = nx + mpx + mna.$$

Transposing,
$$mnx - nx - mpx = mna.$$

Dividing by $mn - n - mp$,
$$x = \frac{mna}{mn - n - mp}.$$

3. The pupil must go through with the whole section in the same manner. And as there are several instances in which the same statement and the same answer will agree with two or more sums, the pupil must tell which they are, and why it so happens.

4. If the teacher should think his pupils need more practice, he may exercise them in the 7th section in the same manner.

The first question may be stated thus,

$$\frac{x+a}{b} = c$$

Multiplying by b ,

$$x+a=bc$$

Transposing,

$$x=bc-a.$$

The 2d will have the following equation.

$$\frac{mx+ma}{n} = b$$

Multiplying by n ,

$$mx+ma=n b$$

Transposing and dividing,

$$x = \frac{nb - ma}{m}.$$

The 3d and 5th will have the following equation.

$$\frac{nx - na}{m} = b.$$

The 4th will be,

$$a - x = \frac{nx}{m}.$$

VIII. DIVISION OF FRACTIONS.

§ 225. We have shown in § 97, that *we divide fractions by dividing the numerator, when the divisor is a factor in the numerator; but if the divisor is not a factor in the numerator, we multiply the denominator by the divisor.*

EXAMPLES.

1. Divide $\frac{3ab}{x}$ by a . Ans. $\frac{3b}{x}$.
2. Divide $\frac{8ax}{bc}$ by b . Ans. $\frac{8ax}{b^2c}$.
3. Divide $\frac{3xy-6x}{2b}$ by $3x$. Ans. $\frac{y-2}{2b}$.
4. Divide $\frac{xy}{z}$ by $-3a$. Ans. $-\frac{xy}{3az}$.
5. Divide $\frac{xyz}{4}$ by $7a^2$.
6. Divide $\frac{60xy}{4z}$ by 7 .
7. Divide $\frac{3ax-6ay}{4axy}$ by $2xy$.
8. Divide $\frac{3ax-y}{a+y}$ by $3ay$.
9. Divide $\frac{4a+27ab}{2x}$ by $6ax$.
10. Divide $\frac{6abx-16axy}{2am+3xy}$ by $2a$.
11. Divide $\frac{14bc^2x+21ac^2}{4abx}$ by $7c^2$.

Question. How do we divide fractions by whole numbers?

12. Divide $\frac{10a^5+5x^5-15ax}{a^4+5x^2-10ax}$ by $5a^5x^4$.

13. Divide $\frac{12ab-14cx}{10ax+16bc}$ by $2ac-4bx$.

14. Divide $\frac{16cd+8x}{x-y}$ by $4cd-cax$.

15. Divide $\frac{9am+12ax}{6mx}$ by $12ax-9a^2$.

16. Divide $\frac{9a^2+21}{14a^3-16}$ by a^3-1 .

IX. FRACTIONS OF FRACTIONS.

§ 226. It was shown, § 98, that a fraction is multiplied by a fraction, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator. Thus, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

EXAMPLES.

1. Multiply $\frac{3a}{5x}$ by $\frac{4b}{c}$. Ans. $\frac{12ab}{5cx}$.

2. Multiply $\frac{9ax}{5b}$ by $\frac{2bx}{3ay}$. Ans. $\frac{18abx^2}{15aby} = \frac{6x^2}{5y}$.

3. Find the product of $\frac{x+y}{3a}$ into $\frac{3y}{x-y}$. Ans. $\frac{xy+y^2}{ax-ay}$.

4. Multiply $\frac{3x}{4}$ and $\frac{4x}{5}$ together.

5. Multiply $\frac{x}{3y}$ into $\frac{6y}{7x}$.

6. Multiply $\frac{3a}{4b}$ into $-\frac{4a}{5}$.

7. Multiply $\frac{x}{a}$ into $\frac{x+y}{x+2y}$.

8. Multiply $-\frac{a}{x}$ into $-\frac{y}{z}$. Ans. $\frac{ay}{xz}$.

9. Multiply $\frac{a}{x}$, $\frac{3x}{y}$, $\frac{4y}{3z}$ together. Ans. $\frac{4a}{z}$.

10. What is the product of $\frac{4ax}{y}$, $\frac{3xy}{2a}$, and $\frac{2}{x}$? Ans. $12x$.

11. What is the product of $\frac{2a}{3b+c}$ into $\frac{2ac-bc}{5ab}$?

12. Multiply $\frac{2am^2-3a^2m}{4ac+2c}$ by $\frac{5am^2}{2am-5c}$.

13. Multiply $\frac{10xy^2+5x^2y}{4x-3y}$ by $\frac{2x^2y-xy^2}{5xy^2+10x^2y}$.

§ 227. When the numerator of one of the factors, and the denominator of the other, can both be divided by the same quantity, it is best to perform that division before the quantities are multiplied. This is done by canceling the quantity that is common to the numerator of one and denominator of the other. Thus, $\frac{ab}{cd} \times \frac{dx}{ay} = \frac{b \times x}{c \times y} = \frac{bx}{cy}$.

14. Multiply $\frac{2x}{5}$ by $\frac{5x^2}{2a}$. Ans. $\frac{x^2}{a}$.

15. What is the product of $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{5ac^2}{2b}$? Ans. $15ax$.

16. Multiply $\frac{15ac+37bc}{10a^2b-2ac}$ by $10ab-2c$. Ans. $\frac{15ac+37bc}{a}$.

17. Multiply $\frac{13ab-13am}{17a^2}$ by $\frac{17a}{b-m}$. Ans. 13 .

18. What is the product of $\frac{a^2-x^2}{a+b} \times \frac{a^2-b^2}{ax+x^2} \times \frac{a^2}{a-x}$? §185 Ans. $\frac{a^2-a^2b}{x}$.

EXERCISES IN EQUATIONS.

§ 228. Generalize the questions in Equations, section 8.

1. The 1st sum on page 64 is performed as follows ;

Stating the question,

$$x = \text{oats.}$$

$$\frac{mx}{n} = \text{barley.}$$

$$\frac{1}{p} \text{ of } \frac{mx}{n} = \frac{mx}{np} = \text{rye.}$$

Forming the equation, $x + \frac{mx}{n} + \frac{mx}{np} = a.$

§ 229. In this and similar equations, we may apply the principle explained in § 88. For, if we multiply by the greatest denominator, np , it is plain that the second term will be $\frac{mnp x}{n}$, which $= mp x$.

Multiplying by np , $np x + mp x + mx = anp$

Dividing by $np + mp + m$, $x = \frac{anp}{np + mp + m}.$

2. The equation in the 2d sum will be $\frac{x}{n} - \frac{x}{mn} = a.$

Multiplying by mn , $mx - x = amn$

Dividing by $m - 1$, $x = \frac{amn}{m - 1}.$

3. The equation in the 3d sum will be

$$x = \frac{x}{m} + \frac{x}{n} - \frac{x}{mn} + a.$$

4. The equation of the 4th sum will be

$$x = \frac{mx}{n} + \frac{mx}{n} - \frac{m^2 x}{n^2} + a.$$

The pupil will now understand how to perform the rest of the section.

The problems under *ratio*, in section 10, page 78, are very good examples for this exercise.

X. UNITING FRACTIONS OF DIFFERENT DENOMINATORS.

§ 230. Before fractional terms can be united, they must be brought to a common denominator, according to the principle explained in § 85. *This is done, as we have shown in § 104, 105, by multiplying each numerator by all the denominators except its own, for new numerators; and all the denominators together for a new denominator.*

EXAMPLES.

1. Unite the following terms, $\frac{2ab}{x} - \frac{ax}{4b} + \frac{am}{bx}$.

Ans. $\frac{8ab^3x - abx^3 + 4abmx}{4b^3x^2} = \frac{8ab^2 - ax^2 + 4am}{4bx}$.

2. Add $\frac{x}{x+1}$ and $\frac{x}{x-1}$ together. § 178. Ans. $\frac{2x^2}{x^2-1}$.

3. Subtract $\frac{3a}{8c}$ from $\frac{4a}{7b}$. Ans. $\frac{32ac - 21ab}{56bc}$.

4. Unite $\frac{x}{x+y} - \frac{y}{z}$. Ans. $\frac{xz - xy - y^2}{xz + yz}$.

5. Unite $\frac{x-y}{2a} - \frac{x+y}{3a}$. Ans. $\frac{ax - 5ay}{6a^2}$.

§ 231. If, in such cases, there is a quantity that is not fractional, we multiply it by all the denominators; and then putting the common denominator under that product, unite it with the other fractions.

6. Unite $a + \frac{b}{c}$. Ans. $a = \frac{ca}{c}$; and then, $\frac{ca}{c} + \frac{b}{c} = \frac{ac+b}{c}$.

Questions. How can fractional terms be united? How can we bring them to a common denominator? Supposing there are quantities connected with them that are not fractional? How do we reduce fractions to the least common denominator?

7. Unite $x + \frac{a+x}{y}$.

Ans. $\frac{xy+a+x}{y}$

8. Unite $a - \frac{a+b}{c}$.

Ans. $\frac{ac-a-b}{c}$.

§ 232. It is generally best to reduce the fractions to the *least* common denominator. This is the *least common multiple* of all the denominators; and is found, when the same quantity is in two or more denominators, by taking that quantity but once in the required product. Thus,

9. Unite $\frac{ar}{5bm^3} + \frac{2c}{5bn} - \frac{3cd}{2m^3n}$


Ans. The common denominator is $5 \times 2 \times b \times m^3 \times n = 10bm^3n$. To obtain this, the first denominator must be multiplied by $2mn$; and therefore the numerator must be multiplied by the same, and becomes $2amnr$. The second denominator must be multiplied by $2m^3$; and therefore its numerator must be multiplied by the same, and becomes $4cm^3$. The third denominator must be multiplied by $5b$; and therefore its numerator must be multiplied by the same, and becomes $15bcd$. The answer is $\frac{2amnr+4cm^3-15bcd}{10bm^3n}$.

10. Unite $\frac{3m^3s}{n^5r^2} + \frac{2ar}{3mn^3r}$.

11. Unite $\frac{15a}{4} + \frac{bc}{2h^3n^3} + \frac{2m^3r}{h^2n^3x}$.

12. Add $\frac{13a^2b-2c}{4ab}$ and $\frac{7ab+8c}{2b+16ab}$

13. Unite $\frac{cde}{21bm^3x^2} - \frac{3cd}{7b^3m^2x}$.

 To know what any denominator must be multiplied by, to make the least common denominator, *divide all the other denominators by such factors of this as will divide them, and then multiply this by the several quotients.*

14. From $\frac{27ad}{2bc^2}$ take $\frac{2abd-3m^2c}{4b^2c^2}$.

15. From $\frac{a+x}{a(a-x)}$ take $\frac{a-x}{a(a \times x)}$.

16. Add $\frac{a+b}{a-b}$ to $\frac{a-b}{a+b}$. Ans. $\frac{2a^2+2b^2}{a^2-b^2} = 2 + \frac{4b^2}{a^2-b^2}$

17. From $\frac{az}{a^2-z^2}$ take $\frac{a-z}{a+z}$. Ans. $\frac{3az-a^2-z^2}{a^2-z^2}$.

18. Unite $\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b}$. Ans. $\frac{a^3+ab^2+b^3}{(a+b)^3}$

19. Unite $x + \frac{a}{b} - ay - \frac{4}{7}$. Ans. $\frac{7bx+7a-7aby-4b}{7b}$

20. Add together $\frac{5a^2+b}{3b}$, and $\frac{4a^2+2b}{5b}$. Ans. $\frac{37a^2+11b}{15b}$.

21. Subtract $\frac{5x+1}{7}$ from $\frac{21x+3}{4}$. Ans. $\frac{127x+17}{28}$.

22. Subtract $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$. Ans. $\frac{4x^2-11x-5}{5x+5}$.

23. Add together $\frac{x}{x-3}$, and $\frac{x}{x+3}$. Ans. $\frac{2x^2}{x^2-9}$.

24. Subtract $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$. Ans. $\frac{4x^2+3}{3x}$.

25. Subtract $\frac{a-b}{a+b}$ from $\frac{a+b}{a-b}$. Ans. $\frac{4ab}{a^2-b^2}$.

26. From $\frac{3a+2b}{c}$, subtract $\frac{5bd-2a-3d}{4cd}$.

27. From $c+2ab-3ac$, subtract $\frac{b^2c-5ab^2c+a^2}{b^2-bc}$.

EXERCISES IN EQUATIONS.

§ 233. Generalize the questions in Equations, section 9, page 70.

XI. DIVISION BY FRACTIONS.

§ 234. Suppose we wish to know how many times $\frac{3}{7}$ is contained in $\frac{6}{7}$. We would divide in the same manner that we follow in dividing 6 pieces by 3 pieces; and say, $\frac{3}{7}$ is contained in $\frac{6}{7}$, *two* times. In the same manner, $\frac{4}{21}$ is contained in $\frac{20}{21}$, *five* times.

The principle is general that *when the divisor and the dividend have a common denominator, the division is performed by dividing the numerator of the dividend by the numerator of the divisor.*

$$\text{Thus, } \frac{a}{b} \div \frac{c}{b} = \frac{a}{c}; \quad \frac{a^3}{x} \div \frac{a^3}{x} = \frac{a^3}{a^3} = a.$$

EXAMPLES.

$$1. \text{ Divide } \frac{4a}{x} \text{ by } \frac{2b}{x}. \quad \text{Ans. } \frac{4a}{2b} = \frac{2a}{b}.$$

$$2. \text{ Divide } \frac{3ab}{cd} \text{ by } \frac{4bx}{cd}. \quad \text{Ans. } \frac{3ab}{4bx} = \frac{3a}{4x}.$$

$$3. \text{ Divide } \frac{2x}{ab} \text{ by } \frac{3xy}{ab}. \quad \text{Ans. } \frac{2}{3y}.$$

$$4. \text{ Divide } \frac{3a-b}{ab} \text{ by } \frac{2a^2}{ab}. \quad \text{Ans. } \frac{3a-b}{2a^2}.$$

$$5. \text{ Divide } \frac{7rx+a^2}{5ast} \text{ by } \frac{4b+ax}{5ast}. \quad \text{Ans. } \frac{7rx+a^2}{4b+ax}.$$

$$6. \text{ Divide } 8a \text{ by } \frac{2ab}{xy}.$$

Explanation. By § 231, $8a = \frac{8axy}{xy}$. Then $\frac{8axy}{xy} \div \frac{2ab}{xy}$
 $= \frac{8axy}{2ab} = \frac{4xy}{b}.$

$$7. \text{ Divide } 3a \text{ by } \frac{4x}{y}. \quad \text{Ans. } \frac{3ay}{4x}.$$

8. Divide ab by $\frac{c}{d}$.

Ans. $\frac{abd}{c}$.

9. Divide $\frac{4a^2x}{y}$ by $\frac{2x^2}{y}$.

Ans. $\frac{2a^2}{x^2}$.

10. Divide $\frac{8mnr}{ax}$ by $\frac{6am^2}{ax}$.

Ans. $\frac{4nr}{3am}$.

11. Divide $\frac{4ay^2x}{bc}$ by $\frac{10ax^2}{bc}$.

Ans. $\frac{2y^2}{5x}$.

12. Divide $\frac{a+bc}{ax}$ by $\frac{4a^2-ab}{ax}$.

Ans. $\frac{a+bc}{4a^2-ab}$.

13. Divide $\frac{am^2}{ab-c}$ by $\frac{ax}{ab-c}$.

Ans. $\frac{m^2}{x}$.

14. Divide $\frac{ax-y}{ab}$ by $\frac{xy}{b}$.

Explanation. In this example, the dividend and divisor has not a common denominator. Our first object, then, is to bring them to a common denominator. This is done by § 230,

$$\frac{ax-y}{ab} \div \frac{xy}{b} = \frac{bax-by}{ab^2} \div \frac{abxy}{ab^2}.$$

$$\text{Ans. } \frac{bax-by}{abxy} = \frac{ax-y}{axy}.$$

15. Divide $\frac{abx}{c}$ by $\frac{dy}{a}$.

$$\text{Operation. } \frac{abx}{c} \div \frac{dy}{a} = \frac{a^2bx}{ac} \div \frac{cdy}{ac} = \frac{a^2bx}{cdy}$$

§ 235. It will be seen that in sums of this kind, after we have brought the terms to a common denominator, the division is performed by putting the numerator of the dividend for the numerator of the answer, and the numerator of the divisor for the denominator of the answer, and make no use at all of the denominators. Let us see then how we obtain these two terms. We multiply the numerator of the dividend by the

denominator of the divisor; and this becomes the *numerator of the answer*. And we multiply the numerator of the divisor by the denominator of the dividend; and this becomes the *denominator of the answer*. By looking at the last two sums, it will be seen that this is the true operation.

§ 236. Hence we obtain the general *rule* for dividing by a fraction. *Multiply the numerator of the dividend by the denominator of the divisor, for a new numerator; and multiply the denominator of the dividend by the numerator of the divisor for a new denominator.*

§ 237. *When the dividend is a whole number, it is changed into a fraction by putting 1 under it for a denominator.* Thus, $x = \frac{x}{1}$; $2a = \frac{2a}{1}$.

EXAMPLES.

$$16. \text{ Divide } \frac{3x}{4} \text{ by } \frac{ax}{2b}. \quad \text{Ans. } \frac{6bx}{4ax} = \frac{3b}{2a}.$$

$$17. \text{ Divide } \frac{a+x}{a-y} \text{ by } \frac{a+y}{a+2x}. \quad \text{Ans. } \frac{a^2+3ax+2x^2}{a^2-y^2}.$$

$$18. \text{ Divide } x^2-2ax+a^2 \text{ by } \frac{1}{x-a}. \quad \text{Ans. } x^3-3x^2a+3a^2x-a^3.$$

§ 238. If both of the numerators, or both of the denominators have the same factor, that factor may be canceled before the operation is performed.

$$\text{Thus, } \frac{ax}{rs} \div \frac{ay}{rt} = \frac{x}{s} \div \frac{y}{t} = \frac{xt}{sy}.$$

$$19. \text{ Divide } \frac{a}{4} \text{ by } \frac{3a}{5}. \quad \text{Ans. } \frac{5}{12}.$$

Questions. What are the rules for dividing by fractions? Explain the rule for fractions that are not of the same denominator. Supposing the dividend is a whole number? In what cases can the work be shortened?

20. Divide $\frac{x-1}{3}$ by $\frac{x+1}{4}$. Ans. $\frac{4x-4}{3x+3}$.

21. Divide $\frac{a+x}{y}$ by $\frac{5a}{4b}$. Ans. $\frac{4ab+4bx}{5ay}$.

22. Divide $\frac{5ab}{4}$ by $-\frac{4a}{y}$. Ans. $-\frac{5by}{16}$.

23. Divide $\frac{a}{a+1}$ by $\frac{3x}{4y}$. Ans. $\frac{4ay}{3ax+3x}$.

24. Divide $x+ax$ by $\frac{3a}{x-y}$. Ans. $\frac{x^2+ax^2-xy-axy}{3a}$.

25. Divide $\frac{y}{y-1}$ by $\frac{y}{3}$. Ans. $\frac{3}{y-1}$.

26. Divide $4a-ay$ by $\frac{4y-ay}{4a}$. Ans. $\frac{16a^2-4a^2y}{4y-ay}$.

27. Divide $\frac{1}{x-y}$ by $\frac{1}{x^2-y^2}$. See § 177, 185. Ans. $x+y$.

28. Divide $\frac{a^2-x^2}{a+b}$ by $\frac{a+x}{(a+b)^2}$.

XII. GENERAL THEORY OF EQUATIONS WITH TWO UNKNOWN QUANTITIES.

§ 239. Every equation of the first degree with two unknown quantities may be reduced so as to be represented by

$$ax+by=c,$$

a designating the algebraic sum by which one unknown quantity is multiplied; and b the sum by which the other unknown quantity is multiplied; and c the algebraic sum of all the known quantities.

For example, the equation $mx+nx+py+qy-ry=l+k$, may be changed into $(m+n)x+(p+q-r)y=(l+k)$, where $(m+n)=a$; $(p+q-r)=b$; and $(l+k)=c$.

§ 240. If there are two equations of this kind, there will be two sets of co-efficients and whole numbers. But for our present purpose it will be best to represent both sets by the same letters; only accompanying the second set with the *accent*, to intimate that they stand for different quantities from the first set. The two equations will then become

$$\begin{aligned} ax + by &= c \\ a'x + b'y &= c'. \end{aligned}$$

§ 241. In order to make the terms of *y* identical, by the rule for common denominators, § 231, we will multiply the first equation by *b'*, and the second by *b*, when we shall have

$$\begin{aligned} ab'x + bb'y &= b'c \\ a'bx + bb'y &= bc' \end{aligned}$$

Subtracting one from the other,

$$(ab' - a'b)x = b'c - bc'$$

Dividing,

$$x = \frac{b'c - bc'}{ab' - a'b}$$

Or, putting the accented quantities last,

$$x = \frac{cb' - bc'}{ab' - ba'}$$

By the same means we shall find $y = \frac{ac' - ca'}{ab' - ba'}$.

§ 242. By the use of accents in the notation of co-efficients, the pupil may readily see the law by which these two formulæ may be found without performing the whole operation of eliminating.

The common denominator is found by forming the two arrangements of the co-efficients AB and BA, with the sign — between them; and then accenting the last letter in each term. Thus, $ab' - ba'$.

To obtain the numerator for answering each unknown quantity, take away from the denominator the letter which designates the co-efficient of that unknown quantity; and put in its place the letter which designates the known quantity, and then accent the last letter in each term.

Thus, first we have for the denominator, $ab - ba$. For the answer to *x*, we will take away the *a*'s and put in *c*'s; and

we shall have $cb - bc$, which, when accented, will be $cb' - bc'$, as above.

For the answer to y , we take away the b 's and put in c 's; and we shall have $ac - ca$, which, when accented, will be $ac' - ca'$.

§ 243. It must be understood that a , b , and c , stand for any algebraic quantities, whether *positive* or *negative*; and whenever those quantities are replaced for a , b , and c , *their own signs* must accompany them in connection with the signs of the above formulas. Thus, let there be two equations:

$$x - 2y = 8$$

$$-3x + 4y = -58$$

then we shall have $a = 1$, $b = -2$; $c = 8$, $a' = -3$, $b' = 4$, $c' = -58$; and by substitution in the formulas,

$$x = \frac{cb' - bc'}{ab' - ba'} = \frac{-8 \times 4 - (-58 \times -2)}{1 \times 4 - (-3 \times -2)} = \frac{32 - 116}{4 - 6} = \frac{-84}{-2} = 42.$$

$$y = \frac{ac' - ca'}{ab' - ba'} = \frac{-58 \times 1 - (8 \times -3)}{4 \times 1 - (-2 \times -3)} = \frac{-58 - (-24)}{4 - 6} = \frac{-34}{-2} = 17.$$

Question 1. Change the above *formula* for the denominator common to their two values into a *rule*.

Question 2. Change the formula for the *numerator* in the value of x , into a rule.

Question 3. Change the formula for the numerator in the value of y , into a rule.

Examples. The pupil may perform a few sums in section 11, page 88, by these rules.

XIII.

INVOLUTION AND POWERS.

§ 244. We have already shown that when a quantity has been multiplied into itself, the product is called the *second power*; and that when it is taken three times as a factor, the product is called the *third power*.^{*} In the same manner, *any number* of times with which a quantity is taken as a factor, will give its name to the product. § 163, 164.

§ 245. In other words, the *exponent* of any quantity represents its name as a power. Thus, a or a^1 is the *first power* of a ; a^2 the *second power* of a ; a^8 the *eighth power* of a ; a^n the *nth power* of a ; $\sqrt[3]{2y}$, or $\sqrt[3]{2y^3}$, or $(2y)^3$, is the *third power* of $2y$.

§ 246. When the *first power* of any quantity is compared with the higher powers, it is called the *root* of those higher powers; because it is from that quantity that they may be said to grow. Thus, x is called the root of x^2 and of x^3 , &c.

§ 247. Involution is finding any power of a quantity. *It is performed by multiplying the quantity into itself, till it is taken as a factor as many times as there are units in the exponent of the required power.*

§ 248. When a quantity is represented by a single letter, the multiplication is performed by simply annexing the required exponent, by § 162. But, if the quantity consists of

Questions. What rule have we for naming powers? How is the name represented on paper? What is called the root of powers? Why? What is involution? How is it performed? How is involution performed on single letters?

* Sometimes, for the sake of conciseness, the second power of a quantity is called the *square* of that quantity; and the third power is called the *cube*.

two or more factors, the exponent must be annexed to each of them; and if there is a numeral factor, the multiplication must be actually performed. Thus, the second power of ax is found to be a^2x^2 ; the third power of $2y$ is 2^3y^3 , or $8y^3$; the fourth power of $3ab$ is $3^4a^4b^4$, or $81a^4b^4$.

EXAMPLES.

- | | |
|----------------------------------|-----------------------------|
| 1. What is equal to $(2ab)^5$? | Ans. $2^5a^5b^5 = 32a^5b^5$ |
| 2. What is equal to $(2ax)^3$? | Ans. $4a^3x^3$. |
| 3. What is equal to $(aby)^3$? | Ans. $a^3b^3y^3$. |
| 4. What is equal to $(5mnx)^4$? | Ans. $625m^4n^4x^4$. |

§ 249. Hence we learn, that *any power of the product of several factors, is equal to the product of their powers*. Thus, the fourth power of ab is $(ab)^4$, or (a^4b^4) . The third power of $16 = 4096$; or it equals $(2 \times 8)^3$ which equals $2^3 \times 8^3$ which $= 8 \times 512 = 4096$.

§ 250. From the preceding remarks, we derive a principle that is of much importance in mathematics. It is the following: $(2a)^2 = 2^2a^2$, or $4a^2$. That is, *four times the second power of any quantity is the same as the second power of twice that quantity*; as, $2^2(x + \frac{3}{2})^2 = 4(x + \frac{3}{2})^2$.

§ 251. We see that when an exponent is to affect only one letter, it is annexed to that letter alone; but when it is to affect a quantity which is represented by more than one letter, that quantity must first be enclosed by a vinculum, or parenthesis, and then the exponent is annexed to that. Thus, the second power of $a+b$ must be represented by either $(a+b)^2$, or $\overline{a+b}^2$, or $\overline{a+b}^2$. The parenthetical form is generally the best.

Questions. How is involution performed if there are more letters than one? What if there is a numeral factor? What does the rule, together with the representation of the power show? What important principle is derived from this? Supposing an exponent is to affect a quantity of more than one letter?

§ 252. In involving compound quantities, it is found best, for most purposes, to give them simply the proper exponent. Thus, the fifth power of $2a-x = (2a-x)^5$. But there are some cases in which it is necessary to perform the multiplication in its extent. And that operation is called *expanding* or *developing* the value of the expression.

5. Thus we expand or develop $(2a-x)^5$, as follows:

$$2a-x$$

$$2a-x$$

$$4a^2-2ax$$

$$-2ax+x^2$$

$$4a^2-4ax+x^2 = (2a-x)^2.$$

$$2a-x$$

$$8a^3-8a^2x+2ax^2$$

$$-4a^2x+4ax^2-x^3$$

$$8a^3-12a^2x+6ax^2-x^3 = (2a-x)^3.$$

$$2a-x$$

$$16a^4-24a^3x+12a^2x^2-2ax^3$$

$$-8a^3x+12a^2x^2-6ax^3+x^4$$

$$16a^4-32a^3x+24a^2x^2-8ax^3+x^4 = (2a-x)^4$$

$$2a-x$$

$$32a^5-64a^4x+48a^3x^2-16a^2x^3+2ax^4$$

$$-16a^4x+32a^3x^2-24a^2x^3+8ax^4-x^5$$

$$32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5 = (2a-x)^5$$

EXAMPLES.

6. Expand the binomial $(a+2x)^4$.

$$\text{Ans. } a^4+8a^3x+24a^2x^2+32ax^3+16x^4.$$

7. Expand the binomial $(2x-3y)^3$.

$$\text{Ans. } 8x^3-36x^2y+54xy^2-27y^3.$$

8. Expand the binomial $(3-x)^4$.

$$\text{Ans. } 81-108x+54x^2-12x^3+x^4.$$


Questions. What are we said to do in performing the *multiplication* in involution? Do we always do that?

9. Expand the trinomial $(a+b-c)^3$.

$$\text{Ans. } a^3 + 2ab - 2ac + b^3 - 2bc + c^3$$

10. Expand the trinomial $(x^2 - 2x + 1)^3$.

$$\text{Ans. } x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$$

 If the binomial has but one letter in each term, the square is immediately known, from § 179 and § 180.

§ 253. The powers of a fraction are found by raising both numerator and denominator to the power required.

$$\text{Thus, } \left(\frac{2}{3}\right)^3 = \frac{2}{3} \text{ of } \frac{2}{3} \text{ of } \frac{2}{3} = \frac{8}{27}; \quad \left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}.$$

EXAMPLES.

$$11. \text{ What is the sixth power of } \frac{a}{b} \quad \text{Ans. } \frac{a^6}{b^6}.$$

$$12. \text{ What is the third power of } \frac{3x}{4y} ? \quad \text{Ans. } \frac{27x^3}{64y^3}.$$

$$13. \text{ What is the sixth power of } \frac{2y}{x} ? \quad \text{Ans. } \frac{64y^6}{x^6}.$$

$$14. \text{ What is the fourth power of } \frac{4y}{5a} ? \quad \text{Ans. } \frac{256y^4}{625a^4}.$$

§ 254. It was shown, § 167, that $a^2 \times a^3 = a^5$; and that $a^2 \times a^3 \times a^3 \times a^3 = a^{12}$. In each of these cases, the *exponent* was added as many times as the quantity was to be taken as a factor; by which we see that *when a quantity has already an exponent, it is raised to any power by multiplying the exponent by the exponent of the required power*. Thus, the fourth power of a^3 is $a^{3 \times 4} = a^{12}$.

§ 255. As to the signs of the powers, by the principle of multiplication, § 172, if the root is *plus* (+), all the powers are *plus*; but if the root is *minus* (—), all the even powers are *plus*, and all the odd powers are *minus*. The second power of $-a$ is $-a \times -a$, which is $+a^2$; the third power is

Questions. How do we find the powers of fractions? How do we involve a quantity that has an exponent? What is the rule for signs of the different powers?

$+a^2 \times -a$, which is $-a^3$; the fourth power is $-a^3 \times -a$, which is $+a^4$; the fifth power is $-a^5$.

EXAMPLES.

15. What is the third power of b^2x^3 ? Ans. b^6x^9 .

16. What is the fourth power of a^3y^2x ? Ans. $a^{12}y^8x^4$.

17. What is the fifth power of bc^3x^2 ? Ans. $b^5c^{15}x^{10}$.

18. What is the third power of ab^2x^2y ? Ans. $a^3b^6x^6y^3$.

19. What is the third power of $-3x^2y^3$? Ans. $-27x^6y^9$.

20. What is the 7th power of $-\frac{3a^2x^3}{5y}$? Ans. $-\frac{2187a^{14}x^{21}}{78125y^7}$.

21. What is the third power of $-\frac{ax^2}{3z}$? Ans. $-\frac{a^3x^6}{27z^3}$.

22. What is the n th power of a^3 ? Ans. a^{3n} .

23. What is the third power of $\frac{2xr^2}{3y}$? Ans. $\frac{8x^3r^6}{27y^3}$.

24. What is the n th power of $\frac{x^2r}{ay^m}$? Ans. $\frac{x^{2n}r^n}{a^ny^{mn}}$.

25. What is the second power of $\frac{-a^3 \times (d+m)}{(x+1)^3}$?
Ans. $\frac{a^6 \times (d+m)^2}{(x+1)^6}$.

26. What is the third power of $-\frac{2}{3}x^2y^3$? Ans. $-\frac{8}{27}x^6y^9$.

§ 256. If we divide any power by its root, we obtain the power next below. Thus, $a^5 \div a = a^4$, or $\frac{a^5}{a} = a^4$. Again, $a^4 \div a = a^3$, or $\frac{a^4}{a} = a^3$; $\frac{a^3}{a} = a^2$; $\frac{a^2}{a} = a$. Let us proceed; $\frac{a}{a} = 1$; but if we diminish the exponent, as we did in the preceding divisions, we have $\frac{a}{a} = a^0$. Hence we learn, that a^0 is equal to 1, however great or small the value of a may be.

Questions. What power of any quantity is equal to 1? How do we learn this?

This may be shown in numbers. 49 is the second power of 7. Now, $49 \div 7 = 7$, the *first* power; and $7 \div 7 = 1$, the *no* power.

§ 257. But we can proceed still farther with our division. $1 \div a = \frac{1}{a}$; $\frac{1}{a} \div a = \frac{1}{a^2}$; $\frac{1}{a^2} \div a = \frac{1}{a^3}$; &c. Or, taking a^0 instead of 1, and dividing by subtracting the exponent of the divisor, § 190, $a^0 \div a = a^{0-1}$ or a^{-1} ; $a^{-1} \div a = a^{-1-1}$ or a^{-2} ; $a^{-2} \div a = a^{-2-1}$ or a^{-3} .

§ 258. Thus, by pursuing two different methods of dividing, we obtain two different sets of expressions, both of which are equal to one another. We will arrange them one under the other, so that any one in the upper line shall be equal to the corresponding one in the lower line.

$$\left\{ \begin{array}{ccccccccccccc} a^5 & a^4 & a^3 & a^2 & a^1 & 1 & \frac{1}{a} & \frac{1}{a^2} & \frac{1}{a^3} & \frac{1}{a^4} & \frac{1}{a^5} \\ a^5 & a^4 & a^3 & a^2 & a^1 & a^0 & a^{-1} & a^{-2} & a^{-3} & a^{-4} & a^{-5} \end{array} \right\}$$

Here, the quantities on the right of 1 in the upper line, and on the right of a^0 in the lower line, are called the *reciprocal powers* of a . They may be read the *first reciprocal power*, the *second reciprocal power*, &c.

§ 259. By this method of notation, we may remove the denominator of any fraction. For, we have only to put it by the side of the numerator, and give it the negative exponent.

Thus, $\frac{a}{x^2} = a \times \frac{1}{x^2} = a \times x^{-2} = ax^{-2}$. So $\frac{a}{x} = \frac{a}{x^1} = a \times \frac{1}{x^1} = ax^{-1}$; where we see that if the denominator has *not* any *written* exponent, 1 is understood, and must be written when it becomes negative, in order to distinguish the quantity from a positive one.

Question. Show the principle by numbers. In how many methods of notation can we continue to divide by the root? What are the corresponding quantities below 1? What are they called? What are we enabled to do by these two methods of notation? How?

EXAMPLES.

Remove the denominators from the following fractions

$$27. \frac{ab}{nx} = abn^{-1}x^{-1}.$$

$$30. \frac{a}{a-b} = a(a-b)^{-1}.$$

$$28. \frac{ax^3}{by^4} = ax^3b^{-1}y^{-4}.$$

$$31. \frac{a+b}{(x+y)^3} = (a+b)(x+y)^{-3}.$$

$$29. \frac{bc}{a^2m^3n^4} = bca^{-2}m^{-3}n^{-4}.$$

$$32. \frac{m}{2-a} = m(2-a)^{-1}.$$

§260. It is very evident that these quantities with negative exponents can be returned to the denominator without affecting their values. And on the same principle, any reciprocal power in the numerator may be removed to the denominator. For, suppose we have the quantity $\frac{ax^{-2}}{by}$. We

know that it is equivalent to $\frac{a}{by} \times x^{-2}$ which $= \frac{a}{by} \times \frac{1}{x^2}$ which $= \frac{a}{byx^2}$. $\therefore \frac{ax^{-2}}{by} = \frac{a}{byx^2}$.

In the same manner, $ab^{-1}c = \frac{ac}{b}$; and $a^{-2}xy^{-1} = \frac{x}{a^2y}$.

§261. Powers with negative exponents are involved in the same manner as if they were positive. Thus, the second power of a^{-2} is $a^{-2 \times 2}$ or a^{-4} ; because the second power of $\frac{1}{a^2}$ is $\frac{1}{a^4} = a^{-4}$. The third power of b^{-2} is b^{-6} , &c.

EXAMPLES.

$$33. \text{What is the fourth power of } a^3b^{-1}? \quad \text{Ans. } a^8b^{-4} = \frac{a^8}{b^4}.$$

$$34. \text{What is the third power of } x^3y^{-2}? \quad \text{Ans. } x^9y^{-6} = \frac{x^9}{y^6}.$$

$$35. \text{What is the } n\text{th power of } b^mx^{-m}? \quad \text{Ans. } \frac{b^{mn}}{x^{rn}}.$$

Question. What if there are reciprocal powers in the numerator? How are reciprocal powers involved?

XIV.

EVOLUTION:

§ 262. Whenever we meet with a quantity that is a power, we know that there must be some other quantity which is the root of that power. See § 244, 246.

The operation of finding the root of any power is called *Evolution*. But as for our purposes, it will be best to subdivide this part of algebra, we shall at present treat of so much of *Evolution* as relates to finding the root by *mere inspection*; and leave that part of it which is generally called *Extraction of Roots* for a future chapter.

§ 263. When a quantity is reduced from its second power to its root, we say we have found its *second root*; when the reduction is from the third power to the root, we say we have found the *third root*, &c.

§ 264. DEFINITION.—*The root of any quantity is a factor which being multiplied into itself a certain number of times will produce that quantity.*

§ 265. As the *second power* is found by *multiplying* the exponent of a root by 2, (§ 254,) so the *second root* will be found by *dividing* the exponent of a power by 2. The third root will be found by dividing the exponent of the power by 3. And, in general, any root may be found by dividing the exponent of the given power by the number expressing the root to be found. Thus, the second root of a^6 is a^3 , because $a^3 \times a^3 = a^6$. The third root of a^6 is a^2 , because $a^2 \times a^2 \times a^2 = a^6$.

Questions. What is Evolution? What is a root of a quantity? How are numeral names given to roots? How do we find the second root of simple quantities?

§ 266. If there are several factors in the quantity, the root of each factor must be taken. The third root of $a^9b^3c^6$ is a^3bc^2 ; for $a^3bc^2 \times a^3bc^2 \times a^3bc^2 = a^9b^3c^6$.

§ 267. If the quantity have a numeral co-efficient, the root of the co-efficient must be taken arithmetically. The second root of $16a^2x^6$ is $4a^1x^3$.

§ 268. The root of a fraction is found by taking the root of both numerator and denominator. The third root of $\frac{8a^6}{27x^9}$ is $\frac{2a^2}{3x^3}$; for $\frac{2a^2}{3x^3} \times \frac{2a^2}{3x^3} \times \frac{2a^2}{3x^3} = \frac{8a^6}{27x^9}$

EXAMPLES.

1. What is the second root of $9a^4$? Ans. $3a^2$.
2. What is the third root of $8x^9$? Ans. $2x$.
3. What is the third root of $64b^9$? Ans. $4b^3$.
4. What is the second root of $\frac{x^2y^4}{a^4c^6}$? Ans. $\frac{xy^2}{a^2c^3}$.
5. What is the second root of $81x^2y^8$? Ans. $9x^1y^4$.
6. What is the third root of $27a^3b^9$? Ans. $3ab^3$.
7. What is the third root of $\frac{8a^6}{27x^9}$? Ans. $\frac{2a^2}{3x^3}$.
8. What is the fourth root of $256a^4x^8$? Ans. $4ax^2$.
9. What is the third root of $\frac{8a^3}{125x^6}$? Ans. $\frac{2a}{5x^2}$.
10. What is the fourth root of $81y^8x^4z^{12}$? Ans. $3y^2xz^3$.
11. What is the second root of $\frac{16a^2b^2}{49x^6y^2}$? Ans. $\frac{4a^1b^1}{7x^3y^1}$.
12. What is the second root of a^2b^{-4} ? Ans. ab^{-2} .
13. What is the third root of $8a^3c^{-9}d^{-3}$? Ans. $2a^1c^{-3}d^{-1}$.
14. What is the third root of $64x^{-3}$? Ans. $4x^{-1} = \frac{4}{x}$.

Questions. Supposing the quantity has several factors? A numeral co-efficient? How for fractions?

§ 269. We may express the division of exponents in the same manner that is used for the division of other algebraical quantities. Thus, the second root of a^4 is $a^{\frac{4}{2}} = a^2$; the third root of a^6 is $a^{\frac{6}{3}} = a^2$; the fourth root of a^4 is $a^{\frac{4}{4}} = a^1$ or a . By the same rule, the second root of a^1 is $a^{\frac{1}{2}}$; and the third root of a^1 is $a^{\frac{1}{3}}$; the second root of a^3 is $a^{\frac{3}{2}}$; the third root of a^3 is $a^{\frac{3}{3}} = a^1$; the n th root of a is $a^{\frac{1}{n}}$; &c.

§ 270. It is on this principle that the exponent $\frac{1}{2}$ is now used as the sign of the second root, and the exponent $\frac{1}{3}$ is the sign of the third root.*


§ 271. Formerly, the sign of a root was indicated by the *radical sign* $\sqrt{}$; and, for some purposes, this sign is still in use.† Whenever it is used, it is placed to the left of the quantity; thus, \sqrt{a} . The *number* of the root is denoted by a little figure placed over the radical sign; unless it is the second root, when the figure 2 is omitted. Thus,

\sqrt{a} is the second or square root of a .

$\sqrt[3]{a}$ is the third or cube root of a .

$\sqrt[n]{a}$ is the n th root of a .

$\sqrt{a+x}$ is the second root of $(a+x)$.

 It must be remembered, that if the radical sign is to effect more than one factor, the vinculum must be used with it; thus, $\sqrt{5a}$, \sqrt{ab} . When the radical sign is placed before a fraction, it must embrace both terms, because the line in the fraction forms a vinculum; thus, $\sqrt{\frac{1}{a}}$ or $\sqrt[1]{\frac{1}{a}}$.

$\sqrt{\frac{ab}{7}}$ is not the same as $\frac{\sqrt{ab}}{7}$. In the first case, it is the

Questions. How may exponents be divided? What signs have been derived from this fact? What other sign for roots? Examples? How is it used with fractions?

* This method was introduced by Simon Stevinus, of Holland, about 1585.

† Invented by Stifelius, in 1544.

square root of the whole fraction; in the last, it is the root of the numerator, divided by 7.

§ 272. With regard to the signs, a *positive even power* is formed from either a positive or negative root; and therefore, *an even root of a positive quantity is either positive or negative*. Thus, the second root of a^2 is either $+a$ or $-a$; because $+a \times +a = a^2$, and $-a \times -a = a^2$. Hence the root is said to be *ambiguous*, and is marked with both signs, $\pm a$. The second root of $2x$ is $\pm \sqrt{2x}$.

§ 273. The odd root of any quantity will have the same sign as that quantity. The third root of $-a^3$ is $-a$.

§ 274. There cannot be an even root of a negative quantity; because no quantity can be multiplied into itself an even number of times, in such a manner as to make a negative power. § 255. See also § 289.

§ 275. There are many quantities whose roots cannot be exactly found. Thus, the second root of 2 can never be found. The root is then expressed either by the radical sign, or a fractional exponent; and called a *surd*, or *irrational quantity*. The second root of 6 is $\sqrt{6}$; the second root of x is \sqrt{x} or $x^{\frac{1}{2}}$; the third root of $a+x$ is $(a+x)^{\frac{1}{3}}$ or $\sqrt[3]{a+x}$.

EXAMPLES.

Express the roots of the following quantities, first by the radical sign, and then by the fractional exponents.

15. The second root of ax ? Ans. \sqrt{ax} or $(ax)^{\frac{1}{2}}$.

16. The third root of $a-y$? Ans. $\sqrt[3]{a-y}$, or $(a-y)^{\frac{1}{3}}$.

17. The second root of a^2x ?
Ans. $\sqrt{a^2x}$, or $a\sqrt{x}$, or $(a^2x)^{\frac{1}{2}}$, or $ax^{\frac{1}{2}}$.

18. The third root of (a^2-a) ; and the third root of x^2 ?

19. The second root of a^3 ; and the second root of ay^3 ?

Questions. Are roots positive or negative? Supposing the name of the root is an odd number? What are surds?

THE SQUARE ROOT OF BINOMIALS.

§ 276. We have shown, § 179, that the second power or square of $a+b$ is $a^2+2ab+b^2$; and that the second power of $a-b$ is $a^2-2ab+b^2$. For distinction's sake we call $a+b$ a BINOMIAL, and $a-b$ a RESIDUAL.*

§ 277. Now, the second power of a *binomial* consists of three terms, the first and last of which are complete powers, and the middle one is twice the product of the roots of the two powers; and all connected together by the sign +. Therefore, whenever we meet with any such quantity, we know immediately *the second root of it is a binomial, and is found by taking the roots of the two terms which are complete powers, and connecting them by the sign +.*

§ 278. The second power of a *residual* is altogether like that of a binomial, excepting the sign before the middle term is —. *The root of such a quantity is taken just as if it were a binomial, only making the connecting sign a — minus instead of a plus.*

EXAMPLES.

1. What is the second root of $a^2+2ax+x^2$? Ans. $a+x$.
2. What is the second root of x^2+2x+1 ? Ans. $x+1$.
3. What is the second root of $4x^2-4ax+a^2$? Ans. $2x-a$.
4. What is the second root of $x^2-x+\frac{1}{4}$? Ans. $x-\frac{1}{2}$.
5. What is the second root of $x^2+xy+\frac{y^2}{4}$? Ans. $x+\frac{y}{2}$.
6. What is the second root of $a+2\sqrt{ab}+b$? Ans. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.
7. What is the second root of $16a^2-16ab+4b^2$?
Ans. $4a-2b$.

Questions. What is the difference between a binomial and residual? How do we know when the second root of a quantity is a binomial? How is it found? How is a residual root found? How can you tell whether a trinomial quantity is a perfect square?

* These terms were introduced by Dr. Recorde, in 1557.

EXTRACTION OF THE SECOND ROOT OF NUMBERS.

§ 279. From the principle last shown, we derive a rule for finding the second root of numbers. For, supposing we have a number of two digits, say 54; it may be represented by $a+b$; that is, 50 may be expressed by a , and 4 by b , so that 50 and $4 = a+b$. We have then the equation $a+b = 54$; a equaling the *tens*, and b equaling the *units*.

Involving both members, we have $a^2 + 2ab + b^2 = 2916$.

§ 280. Now, if any of my pupils should see this last equation without being told what was the *root* we first employed, he would immediately know, by § 277, that the second root of the first member is $a+b$, and of course, the root of 2916 will also consist of two quantities that may be represented by a and b . Thus, $\sqrt{2916} = a+b$.

It would be only necessary to inform him that a stands for an even number of *tens*, and b stands for a quantity *less* than ten.

§ 281. As it is known that the second power of 10 is 100, the second power of 20 is 400, the second power of 80 is 6400, &c.; it is evident that a (which stands for *tens*) must be the root of the *hundreds*. Therefore, to find a , is simply to find what is the greatest second power of 29 in the *hundred*, and then to take the root of it. The second power is 25 *hundred*, and the root is 5; for 6 times 6 = 36, which is too much. But the root of hundreds is in tens; therefore, $a = 50$, and $a^2 = 2500$.

Questions. From what principle do we derive a rule for extracting the second root of numbers? In what manner may a number be similar to a binomial? Which letter represents the tens? How many figures in the *power* are represented by b ? How can we find the second root of a number represented by a^2 ? And then the value of a ? Then what part of the binomial square is left?

§ 282. Taking a^2 from the first member of the above equation, and 2500 from the second, we have left the equation

$$2ab + b^2 = 2916 - 2500 = 416.$$

As $a = 50$, $2a = 100$; hence, substituting the value of $2a$, we have $100b + b^2 = 416$. And, of course, if we divide 416 by 100, we shall find *very nearly* the value of b . $416 \div 100 = 4$, which we will *suppose* to be the value of b .

§ 283. Let us see whether b does actually equal 4. Our last equation was

$$100b + b^2 = 416$$

But $100b + b^2 = (100 + b) \times b$. See § 196.

Substituting $(100 + b) \times b$, for $100b + b^2$, $(100 + b) \times b = 416$.

If $b = 4$, then $(100 + 4) \times 4$ should = 416.

which is the fact. And therefore, b does equal 4. Hence, $a + b = 50 + 4 = 54$, the second root of 2916.

§ 284. We have now algebraically discovered the following *rule for extracting the second root* of any number that consists of not more than four figures. *Find the greatest second power of hundreds, and put its root in the answer. Subtract that second power from the whole number, and divide the remainder by twice the root already found. Suppose the quotient to be the number that completes the root; add it to the root already found, and also to the last divisor. Multiply the divisor thus augmented, by the quotient figure last found, and subtract the product from that remainder which was divided. If there is nothing left, you have found the true root.*

§ 285. It sometimes happens that the product which is obtained by multiplying the augmented divisor, is greater than the number which was divided. In such cases, the figure last put in the root is too large; and therefore it must be erased, and a smaller figure substituted.

Questions. Of which factor can we find the value? How? What can we do with this value? What is the rule for extracting the root as now explained? What little alteration is sometimes required?

EXAMPLES.

1. What is the second root of 7396?

Ans. a^2 must equal 6400; and $a^1 = 80$.
Therefore the remainder $2ab + b^2 = 996$.
 $2a = 160$, which is contained in 996, 6
times; for b times $(2a + b)$, or 6 times
 $160 + 6 = 996$. Therefore the second root
of 7396 = 86.

$$\begin{array}{r} 7396 \overline{)80} \\ 6400 \\ \hline 160 \overline{)996} \overline{)6} \\ 166 \overline{)996} \end{array}$$

2. What is the second root of 4624? Ans. 68.

3. What is the second root of 5184? Ans. 72.

4. What is the second root of 529? Ans. 23.

5. What is the second root of 1764? Ans. 42.

6. What is the second root of 9604? Ans. 98.

7. What is the second root of 2304? Ans. 48.

§ 286. It will be found by trial, that if a number is in *units*, its second power will be in *tens*; for the square of 9 = 81. If the number is in *tens*, its second power will contain *more than two* figures, and *less than five*; for it will take the square of 100 to make 10000. The principle is general, *that every figure in the root except the left-hand one, requires two figures in the second power; and for the left-hand figure, there may be either one or two in the power.*

Hence, by pointing off the number from the right, in periods of two, (as in the next example,) we may learn how many figures will be in the root.

§ 287. When it is found that the root will consist of more than two figures, we are to suppose at first that a^2 is represented by the *first* period; and after the first figure in the root is found, the first two periods represent a^2 ; and so period

Questions. How many figures in the power belong to each figure in the root? How then shall we learn the number of figures in the root? How is the operation performed when the root contains more than two figures?

After period, a represents that part of the root which is found, and b embraces all which *remains* to be found.

8. What is the second root of 55225?

Operation. At first, $a = 200$, and $b = 30$. Afterwards, $a = 230$, and $b = 5$.

$$\begin{array}{r} 55225(200 \\ 40000 \\ \hline \frac{400}{30} \left\{ \begin{array}{l} 15225(30 \\ 430 \end{array} \right\} 12900 \\ \hline \frac{460}{5} \left\{ \begin{array}{l} 2325(5 \\ 465 \end{array} \right\} 2325 \\ \hline \end{array}$$

§ 288. The operation may be abbreviated by omitting the ciphers in the roots and powers, as in the margin. When we do this, however, we must remember that a represents an order *ten times* as great as that represented by b ; and therefore, when we double it for a divisor, we must *annex a cipher* to it, to bring it to the same denomination as b .

$$\begin{array}{r} 55225(235 \\ 4 \\ \hline \frac{40}{3} \left\{ \begin{array}{l} 152 \\ 43 \end{array} \right\} 129 \\ \hline \frac{460}{5} \left\{ \begin{array}{l} 2325 \\ 465 \end{array} \right\} 2325 \\ \hline 0 \end{array}$$

9. What is the second root of 2125764? Ans. 1458.
 10. What is the second root of 20736? Ans. 144.
 11. What is the second root of 10342656? Ans. 3216.
 12. What is the second root of 36372961? Ans. 6031.

§ 289. If the number to be evolved is a vulgar fraction, it must be reduced to decimals, which are to be pointed off from units towards the right, instead of towards the left.

Thus, $\sqrt{\frac{7}{8}} = \sqrt{.25} = .5$ $\sqrt{\frac{3}{7}} = \sqrt{.428571428}$, &c.

It will be pointed off thus, $.42857142$, &c. The pupil will find the answer to be $.65465+$.

Question. Suppose the power is a vulgar fraction?

SECTION XV.

PURE QUADRATIC EQUATIONS.

§ 290. There are several kinds of equations. We have hitherto attended to that kind which contains an unknown quantity in the *first degree* only. But there are equations that contain an unknown quantity in the *second* power; and also in the *third* power, and so on indefinitely.

§ 291. In order to distinguish the different kinds of equations, we are accustomed to call those which contain only the *first* power of the unknown quantity, *simple* equations, or equations of the *first degree*. Those which contain the *second* power of the unknown quantity, are called *quadratic* equations, or equations of the *second degree*. Those which contain the *third* power of the unknown quantity, are called equations of the *third degree*, &c.

§ 292. There are two kinds of quadratic equations. Those which contain *only* the second power of the unknown quantity, are called *pure* quadratic equations; those which contain *both* the *second* and the *first* power of the unknown quantity, are called *affected* quadratic equations.*

§ 293. *A pure quadratic equation is reduced by simply extracting the root of both members.*

Questions. How do we distinguish different kinds of equations? How many kinds of quadratics? How do we reduce a pure quadratic equation?

* All pure equations are called by some algebraists simple equations.

EQUATIONS.—SECTION 15.

1. There is a number, such, that by adding 5 to it for one factor, and subtracting 5 from it for another factor, we may obtain 96 for the product. What is that number?

Stating the question,

Let x = the number.

$x+5$ = one factor.

$x-5$ = the other.

$(x+5) \times (x-5)$ = the product

Forming the equation,

$(x+5) \times (x-5) = 96$

Multiplying, § 172,


$x^2 - 25 = 96$

Transposing and uniting,

$x^2 = 121$

Extracting the root,

$x = \pm 11.$

 In this example, we have given 11 the uncertain sign, because 121 is the second power of either $+11$ or -11 . But the x^2 we know to be the second power of $+x$, because x was $+$ in the supposition. We have therefore found that the required number is either 11, or 11 *less than nothing*. Both values will satisfy the conditions of the question. If $11 = x$, then $11+5 = 16$, and $11-5 = 6$. And $16 \times 6 = 96$. If $-11 = x$, then $-11+5 = 6$, and $-11-5 = -16$. And $6 \times -16 = -96$.

§ 294. In this manner, every quadratic equation will have two answers; but the conditions of the question will generally determine which answer is the one required. In pure quadratics, we generally suppose the positive answer to be the true one.

2. What two numbers are those which are to one another as 3 to 5; and whose squares, added together, make 1666?

Questions. Why \pm in the answer? How many answers then in quadratics?

Stating the question,

$x =$ the greatest.

$\frac{3x}{5} =$ the least.

$x^2 =$ square of the greatest.

$\frac{9x^2}{25} =$ square of the least.

Forming the equation,

$$x^2 + \frac{9x^2}{25} = 1666$$

Multiplying and uniting,

$$34x^2 = 41650$$

Dividing by 34,

$$x^2 = 1225$$


Extracting the root,

$$x = \pm 35 \text{ the greatest.}$$

$$\frac{3}{5} \text{ of } 35 = 21 \text{ the least.}$$

3. The distance to a certain place is such, that if 96 be subtracted from the square of the number of miles, the remainder will be 48. What is the distance? Ans. 12 miles.

4. There is a field containing 108 square rods; and the sum of the length and breadth is equal to twice their difference. Required the length and breadth?

 In stating the question, $x =$ the length, and $\frac{108}{x} =$ the breadth; because the two multiplied together make 108, the area. In this operation, there will be an equation having $-x^2$ for one member. But as by § 274, we cannot find the second root of a *negative* quantity, we must first change the signs of all the terms. Ans. Length, 18; breadth, 6.

5. There is a rectangular field, whose breadth is $\frac{5}{6}$ of the length. After laying out $\frac{1}{6}$ of the whole ground for a garden, it was found that there were left 625 square rods for mowing. Required the length and breadth of the field?

Ans. Length, 30 rods; breadth, 25.

6. Two men talking of their ages, one said that he was 94 years old. Then, replied the younger, the sum of your age and mine, multiplied by the difference between our ages, will produce 8512. What is the age of the younger?

Ans. 18 years.

7. If three times the square of a certain number be divided by 4, and if the quotient be diminished by 12, the remainder will be 180. What is the number? Ans. 16.

8. A person bought a quantity of sugar at such a rate, that the *price* of a pound was to the number of pounds as 4 to 5. If the cost of the whole had been 45 cents more, the *number of pounds* would have been to the price of a pound as 4 to 5. How many pounds did he buy, and what was the price per pound?

Operation.—Let x = number of pounds, and y = the price, and xy = the whole cost. Then, by the first condition, $y = \frac{4x}{5}$. If x pounds cost $xy + 45$, then one pound would cost $\frac{xy + 45}{x}$. Therefore, by the second condition, $x = \frac{4xy + 180}{5x}$; or, $5x^2 = 4xy + 180$. But $y = \frac{4x}{5}$; $\therefore 4xy = \frac{16x^2}{5}$. Whence the equation $5x^2 = \frac{16x^2}{5} + 180$.

Ans. 10 pounds, at 8 cts. a pound.

§ 295. The pupil will find that whenever both unknown quantities are found in one term, the second method of extermination (§ 131) is the best.

9. There are two numbers whose product is 144, and the quotient of the greater by the less is 16. What are the numbers? Ans. 48 and 3.

10. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132? Ans. 14 and 8.

SECTION XVI.

AFFECTED QUADRATIC EQUATIONS.

§ 296. It is but very rarely that we find quadratic equations so simple as those in the preceding section. Most generally, when reduced to their simplest terms, they contain the *second power* of the unknown quantity in one term, the *first power* of the unknown quantity in another term, and a known quantity as a third term; thus, $x^2 + 2x = 24$.

§ 297. Such equations are called *affected quadratic equations*, because the operation on the second power is affected by a different power of the same quantity.*

§ 298. It is of no importance how many terms are found in an equation, provided they are the first and second powers of the unknown quantity, and known numbers; because they can all be united so as to form but three. Thus, $x + 8 - 2x^2 + 3x + 10 = 5x^2$, can be united so as to form $-7x^2 + 4x = -18$; or $7x^2 - 4x = 18$.

§ 299. Equations of this kind would seem at first to be very difficult of solution; but by the application of a principle which is found in raising a binomial to its second power, and evolving it again, we manage them with very little trouble.

§ 300. Our object, then, in the first place, is to form from these two terms which contain the unknown quantity, a complete second power of some binomial. Of what quantities this binomial is composed, we care not; say $x^2 + 2ax + a^2$.

Questions. What is the most general form of quadratics? What are such equations called? How many terms may they contain? What is the first procedure in solving such equations?

* This name was introduced by Vieta, about the year 1600. It is derived from the Latin *affecto*, to pester or trouble.

§ 301. Now, the second power of a binomial, (see § 179,) has in its *first term*, the second power of the first term in the binomial. Thus,

$$\text{Binomial,} \quad x+a.$$

$$\text{Second power, } x^2+2ax+a^2.$$

For this, we use the second power of the unknown quantity in the equation, say x^2 .

§ 302. Again, the second power of the binomial has in its *second term*, the first power of the first term in the binomial *as one factor*; and twice the last term of the binomial *as the other factor*. That is, it has $2a$ times x . Thus,

$$\text{Binomial,} \quad x+a$$


$$\text{Second power, } x^2+2ax+a^2.$$

For this, in our equation, we use the term that has x the *first power*, in it. And we also determine, that whatever its co-efficient is, *that co-efficient shall represent the value of $2a$* .

§ 303. Finally, the *last term* in the second power of the binomial is the second power of *half that other factor* which accompanies x in the *second term*; that is, it is the second power of once a . Therefore, in our equation, whatever was the co-efficient in the second term, *the second power of half that co-efficient will be the last term* which it is necessary to add, in order to make up the complete second power of a binomial.

EXAMPLES.

1. Suppose we have the equation $x^2+4x=96$.

 We are first to make the left-hand member similar to the formula $x^2+2ax+a^2$. The co-efficient of the second term in the question is 4; and therefore 4 represents the $2a$ in the formula. The half of 4, which is 2, will represent a : and

Questions. In this operation, what part of the binomial square do we make of x^2 ? What do we consider as represented by the co-efficient of x in the second term? Why? How do we obtain the last term of the binomial square? Why is this correct?

the second power of 2, which is 4, will represent a^2 . Therefore, from the left-hand member of the equation we may form the complete second power x^2+4x+4 . But if we add 4 to this member, we must also add 4 to the right-hand member, to preserve the equation. We shall then have the equation

$$x^2+4x+4=96+4$$

Or, $x^2+4x+4=100$

Extracting both sides, § 277, 284, $x+2=\pm 10$

Transposing 2, $x=8$ or -12 .

Proof. If $x=8$, then $x^2+4x=64+32=96$.

If $x=-12$, then $x^2+4x=144-48=96$. For, it must be remembered that if x is -12 , then $4x=-48$, which is to be added to 144; and adding -48 , is done by writing it immediately after the 144 with its own sign.

2. Given $x^2+6x=16$, to find the value of x .

Here, $6=2a$ in the formula. $\therefore a=3$, and $a^2=9$. Therefore, completing the square, $x^2+6x+9=16+9=25$.

Extracting the root, $x+3=\pm 5$

Transposing, $x=2$ or -8 .

3. Given $x^2+8x=33$.

Half the co-efficient of the second term is 4; and the second power of 4 is 16. Therefore, the square is completed thus, $x^2+8x+16=33+16$.

Ans. $x=3$ or -11 .

4. Given $x^2+4x=32$, to complete the square.

Ans. Half the co-efficient of the second term is 2; and the second power of 2 is 4. Therefore, the square, when completed, is $x^2+4x+4=32+4$. And $x=4$ or -8 .

5. Given $x^2-4x=32$, to complete the square.

Operation.—The left-hand member of this equation is a part of the second power of a *residual* quantity. See § 276. Now, as the second power of a residual is the same as that of a binomial, with the exception of the second term being —

instead of $+$, the square is completed in the same manner as before. Thus, half of the co-efficient is 2, and the second power of 2 is 4. Therefore, when the square is completed, it is

$$x^2 - 4x + 4 = 32 + 4 = 36$$

Extracting the root, see § 278,

$$x - 2 = \pm 6$$

And transposing,

$$x = 8 \text{ or } -4.$$

6. Given $x^2 - 6x = -8$, to find x .

Completing the square,

$$x^2 - 6x + 9 = -8 + 9 = 1$$

Extracting the root,

$$x - 3 = \pm 1$$

Transposing,

$$x = 2 \text{ or } 4$$

Complete the square in the following equations, and find the value of x .

7. $x^2 = 4x = 140.$

Ans. $x = 10 \text{ or } -14.$

8. $x^2 + 8x = 65.$

Ans. $x = 5 \text{ or } -13.$

9. $x^2 - 4x = 45.$

Ans. $x = 9 \text{ or } -5.$

10. $x^2 - 15x = -54.$

The half of 15 is $\frac{15}{2}$; and

completing the square, $x^2 - 15x + \frac{225}{4} = -54 + \frac{225}{4}.$

Or,

$$x^2 - 15x + \frac{225}{4} = -\frac{216}{4} + \frac{225}{4} = \frac{9}{4}.$$

Extracting the root,

$$x - \frac{15}{2} = \pm \frac{3}{2}$$

Transposing,

$$x = 9 \text{ or } 6.$$

11. $x^2 + x = 42.$ In this example, half of the co-efficient is $\frac{1}{2}$. Ans. $x^2 + x + \frac{1}{4} = 42\frac{1}{4}.$ And $x = 6 \text{ or } -7.$

12. $2x^2 + 16x = 40.$

We must remember that the first term in the second power of the binomial is x^2 . On this account, if x^2 in the equation has a co-efficient, we must divide the equation by that co-efficient. This example then becomes

$$x^2 + 8x = 20.$$

Ans. $x = 2 \text{ or } -10.$

13. $5x^2 - 60x = 800.$

Ans. $x = 20 \text{ or } -8$

14. $8x^2 + 8x = 16$. (See Ex. 11.) Ans. $x = 1$ or -2 .

15. $6x^2 - 5x = 1$. When divided, it is $x^2 - \frac{5}{6}x = \frac{1}{6}$.

Half of $\frac{5}{6} = \frac{5}{12}$, which squared $= \frac{25}{144}$.

Completing the square, $x^2 - \frac{5}{6}x + \frac{25}{144} = \frac{24}{144} + \frac{25}{144} = \frac{49}{144}$.

Extracting the root, $x - \frac{5}{12} = \pm \frac{7}{12}$.

Transposing, $x = 1$, or $-\frac{2}{12}$.

16. $\frac{x^2}{2} - \frac{x}{3} = \frac{133}{6}$. Multiplied, it is $x^2 - \frac{2x}{3} = \frac{133}{3}$.

Completing the square, $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{133}{3} + \frac{1}{9} = \frac{400}{9}$.

Extracting the root, $x - \frac{1}{3} = \frac{20}{3}$.

Transposing, $x = 7$ or $-6\frac{1}{3}$.

17. $3x^2 - 3x = -\frac{2}{3}$. Divided, it is $x^2 - x = -\frac{2}{9}$. And $x = \frac{2}{3}$ or $\frac{1}{3}$.

18. $14x - x^2 = 45$. Transposed, it is $-x^2 + 14x = 45$.

The first term of the second power of a binomial is positive. We must, therefore, change the signs of the whole equation, § 62, in order to make the first term of the equation positive. It will then be $x^2 - 14x = -45$. Ans. $x = 9$ or 5 .


In review of the foregoing principles, we establish the following rule.

RULE FOR THE SOLUTION OF QUADRATIC EQUATIONS.

§ 304. *Unite and transpose the terms of the equation in such a manner that the second power of x shall occupy the first term of the equation, and be positive. Let the first power of x be in the second term, and all the known quantities be in the right-hand member.*

§ 305. *Then, if the first term has any co-efficient, divide the whole equation by that co-efficient.*

To both sides of the equation, add the second power of half the co-efficient of the second term; and the unknown side will then be a complete square. Extract the second root of each side, taking care to prefix the double sign \pm to the right-hand member. And then, by transposition, the value of the unknown quantity is easily found.

 In this kind of equations, it is not necessary to destroy the fractions. Indeed we are sometimes obliged to make them.

EQUATIONS.—SECTION 17.

1. What two numbers are those whose sum is 12, and whose product is 35?

Stating the question,

x = the greater number.

$12 - x$ = the less.

$12x - x^2$ = the product.

Forming the equation,

$12x - x^2 = 35,$

Transposing, § 304,

$x^2 - 12x = -35,$

Completing the square, $x^2 - 12x + 36 = -35 + 36 = 1,$


Extracting the root,

$x - 6 = \pm 1,$

Transposing,

$x = 7$ or $5.$

2. Find two numbers, such, that their difference shall be 7, and their product 240.

 The numbers are x and $x + 8.$ Ans. 12 and 20.

3. The difference of two numbers is 4, and their product

96. What are those numbers? Ans. 12 and 8.

4. There are two numbers, whose difference is 9; and whose sum, multiplied by the greater, produces 266. What are those numbers? Ans. 5 and 14; or $-9\frac{1}{2}$ and $-18\frac{1}{2}.$

Questions. What is the first step in the solution of affected quadratic equations? What if the first term has a co-efficient? How do we complete the square? What is the next step? What are we to do in case of fractions?

5. Divide the number 56 into two such parts, that their product shall be 640. Ans. 40 and 16.

6. Divide the number 60 into two such parts, that their product shall be 864. Ans. 36 and 24.

7. The difference of two numbers is 6, and the sum of their squares is 50. What are the numbers? Ans. 7 and 1.

8. The ages of a man and his wife amount to 42 years, and the product of their ages is 432. What is the age of each? Ans. Man's, 24 years; wife's, 18 years.


9. A merchant has a piece of broadcloth and a piece of silk. The number of yards in both is 110; and if the square of the number of yards of silk be subtracted from 80 times the number of yards of broadcloth, the difference will be 400. How many yards are there in each piece?

Ans. 60 of silk; 50 of broadcloth.

10. A is 4 years older than B; and the sum of the squares of their ages is 976. What are their ages?

Ans. A's age, 24 years; B's, 20 years.

11. A nursery-man planted 8400 trees at equal distances, in the form of a rectangle, having 50 trees more in front than in depth. What was the number in front?

 Let x = the length, and $x - 50$ = the breadth.

Ans. 120 trees.

12. Divide the number 30 into two such parts, that their product may be equal to eight times their difference.

Ans. 6 and 24.

13. A person being asked his age, answered, My mother was 20 years old when I was born, and her age multiplied by mine exceeds our united ages by 2500. What was his age?


Ans. 42 years.

14. The length of a certain field exceeds its width by 8 rods; and its area is 768 rods. What are the dimensions of the field?

Ans. Length, 32 rods; width, 24 rods.


15. Divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5. Ans. 20 and 40.

16. Each of two captains distributes 1200 dollars equally among his men. A has 40 men more than B, and B's men receive five dollars apiece more than A's. How many men had each captain?

 If x = the number of B's men, then $\frac{1200}{x}$ = the share of each of them. Also, $x+40$ = number of A's men, and $\frac{1200}{x+40}$ = the share of each of them. Ans. A had 120; B, 80.

17. A man bought a certain number of sheep for 80 dollars. If he had bought four more sheep for the same money, they would have come to him 1 dollar a piece cheaper. How many sheep did he buy? Ans. 16 sheep.

18. A merchant sold a quantity of cloth for \$56, by which he gained as much per cent. as the whole cost him. How much did it cost him?

 If x = cost, then $56-x$ = gain. *Per cent.* is always written as the 100th of a number. In this case, it is $\frac{x}{100}$, and the cost multiplied by it makes the gain. Ans. \$40.

19. A person buys a horse for a certain sum, and afterwards sells him for \$144, and gains exactly as much per cent. as the horse cost. What did the horse cost him? Ans. \$80.

20. A person bought two pieces of cloth of different sorts: of which the finer cost 4s. a yard more than the other. For the finer he paid £18, but for the coarser, which exceeded the finer in length by 2 yards, he paid only £16. How many yards were there in each piece?

Ans. 18 yards of the finer: 20 of the coarser.





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